Signal-to-interference percolation in Poisson–Voronoi cities

H. P. Keeler *[1], B. Błaszczyszyn †[2], and E. Cali ‡[3]

[1]University of Melbourne, Australia
[2]Inria/ENS, Paris, France
[3]Retired, Paris, France

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Abstract

We investigate connectivity in wireless device-to-device networks in urban environments via a percolation model on a Poisson–Voronoi graph. Edges are formed according to both a signal-to-interference ratio criterion and a graph-based line-of-sight constraint, thereby unifying two mechanisms previously studied separately in continuum percolation by Dousse et al. (2006) and Le Gall et al. (2021).

The Poisson–Voronoi tessellation models the street layout, with users located along streets and relays at crossroads, yielding a Cox point process. We show that percolation fails at low user intensity, occurs at intermediate intensities under low interference, and breaks down again at high user intensity, exhibiting a non-monotonic dependence on density due to the competing effects of connectivity and interference.

Our main technical contribution is a new stabilization radius for this Poisson-Voronoi percolation model—building on the concept introduced by Hirsch et al. (2019)—extending the nearest-seed stabilization of Gilbert-type models to account for interference from adjacent streets, potentially involving multiple Voronoi seeds at each crossroads. Quantitative aspects of the model were explored numerically in Keeler et al. (2023).

1 Introduction

Picture a city with phone relay stations, located at various crossroads (or street intersections), and, scattered along each street, people willing to relay data through their phones and other devices, forming a large *device-to-device network*. What is the probability that data can be relayed through such a network? For large networks, these types of questions motivate the field of probability known as *percolation theory*, which has its historical origins in studying physical materials and wireless networks.

We present a model that brings together two respective continuum percolation models developed over two distinct research paths. The first path introduced a seminal wireless network model that started with the paper by Dousse, Baccelli, and Thiran [10] and culminated with the paper by Dousse and co-authors [11]. This work established a continuous model of a wireless network in which the connectivity mechanism is based on *signal-to-interference*. Significantly, the work proved that percolation occurs in this wireless network model such that there exists a critical threshold that the intensity of wireless network nodes must exceed for the network to percolate. Furthermore, the work proved the crucial fact that there exists a *second* critical intensity threshold, where percolation stops if the intensity exceeds this second threshold, resulting in a percolation island of sorts.

The second path goes via a paper by Le Gall, Błaszczyszyn, Cali and En-Najjary [23] who recently proposed a novel percolation model to study the communication abilities of wireless networks known as device-to-device networks. In such network models, it is usually assumed that the user devices, which are typically mobile (or cellular) phones, can directly communicate with other devices located within a certain

*h.keeler@unimelb.edu.au
†Bartek.Blaszczyszyn@ens.f
‡elie.cali@orange.com

range, but the devices can also communicate with nearby transmitting-receiving relays in the city. More precisely, their percolation model [23] uses a connectivity (or communication) model that is purely geometric, where user devices and relays can only communicate with each other if they are within some fixed distance of each other and they are located on the same street. The first connectivity requirement is known as the *Gilbert model*, which stems from a classic model in (continuum) percolation theory in the pioneering work by Gilbert [14]. The second connectivity requirement is known as a *line-of-sight model*, a self-explanatory term coming from a discrete percolation model by Bollobás, Janson and Riordan [6].

Le Gall et al. [23] introduced a percolation model based on a Cox point process by first constructing a Voronoi tessellation from a Poisson point process. Using the corresponding Voronoi graph for the street layout, user devices (or users) are scattered independently on each edge (that is, street) according to a Poisson point process, whereas a relay is placed independently at each vertex (that is, crossroads or intersection) according to a Bernoulli trial.

To show percolation under the Cox line-of-sight model, our proof techniques hinge upon proving that the percolation model satisfies two key properties, which are familiar in the Cox percolation literature. The first property is that the underlying random street model is *stabilizing* for a specific *stabilization radius*, which means that the street model's spatial dependence weakens sufficiently over large distances. The second property is that the stabilizing street model is *asymptotically essentially connected*, which, loosely speaking, means that if there exist two disconnected graph components in a bounded observation window, we can always enlarge the window by a fixed (finite) amount so that the two components become connected.

1.1 Contributions

We enhance the line-of-sight model in the paper by Le Gall, Błaszczyszyn, Cali, and En-Najjary [23]. Instead of using the Gilbert connectivity model, we introduce a more general connectivity model based on signal-to-interference. This is a fundamental concept in information theory that sets an upper limit on the capacity for successful communication, making it the core component in wireless network models [10, 11, 12]. In everyday language, this concept says that in a room full of people trying to speak to you, it is not simply the distance and volume of a single speaker that dictates your ability to understand them, but rather their distance and volume compared to the total interference of everyone else trying to speak to you. Arguably, the signal-to-interference approach is a more natural model for wireless communications.

In the current work, we replace the fixed distance (Gilbert) requirement with a more realistic signal-to-interference requirement, and then study the resulting percolation model. This gives a new model, for which we then present three key results stating when the connectivity graph percolates and does not percolate. In particular, we give a new result proving the existence of a threshold for the user intensity above which percolation ceases to occur.

A key technical contribution of this work is a new stabilization radius required to handle the spatial dependencies introduced by the signal-to-interference model. The stabilization framework of Hirsch, Jahnel and Cali [16] and the Gilbert percolation work of Le Gall and co-authors [23] both used a stabilization radius $R^*(x)$ defined as the distance from a point x to the nearest seed point of the underlying Poisson point process generating the Voronoi tessellation. This radius suffices when connectivity depends only on geometric distance, because the spatial dependence at a point x is determined solely by nearby streets.

However, under the signal-to-interference model, connectivity on a street depends not only on users along that street but also on interference from users on adjacent streets forming the street neighborhood. This additional dependence requires a larger stabilization radius $R(x)$ that accounts for the neighborhood structure. Specifically, for vertices (crossroads) $x \in V$, we define $R(x) = \sup_{e:x\in e} R_2(e, x)$ where $R_2(e, x)$ captures the maximum distance from x to seed points generating the entire neighborhood of edge e, involving up to eight distinct seed points with probability one. For non-vertex points, we retain $R(x) = R^*(x)$. This construction ensures $R(x) \geq R^*(x)$ for all points, with strict inequality at vertices where interference from adjacent streets matters. We prove that this new radius satisfies the required stabilization properties.

1.2 Related work

Gilbert created the field of continuum percolation with his pioneering paper [14] to study wireless networks by introducing a novel random spatial model. The model uses a Poisson point process for the locations of transmitter-receivers, where any two are connected only if they are located within a fixed distance of each other. Gilbert constructed an argument based on branching processes and showed percolation was possible in his random spatial model. The Poisson case has been extensively studied over the years; see the monograph by Meester and Roy [27]. Researchers have also examined percolation models with the Gilbert connectivity requirement using sub-Poisson [4, 5], Ginibre [13], and Gibbsian point processes [18, 30].

Using a Poisson point process, Dousse, Baccelli, and Thiran [10] did the first work to incorporate interference (and the ability to reduce it) into a percolation model, which was developed further mathematically in a subsequent paper [11]. These papers [10, 11] demonstrated and proved that a trade-off between interference factor and network density exists. Meester and Franceschetti [12] gave a full account and good introduction of this work in their monograph on signal-to-interference percolation.

Our percolation model is based on a Poisson–Voronoi tessellation, which is a central object of study in stochastic geometry [25, 28, 29]. In the context of wireless networks, Baccelli and Błaszczyszyn [1] proposed and examined an early (random coverage) model based on the Poisson-Voronoi tessellation.

The continuum percolation model presented here is based on a Cox point process, which effectively adds a layer of randomness to the classical Poisson model. Hirsch, Jahnel and Cali [16] introduced a general framework requiring the two aforementioned properties in a Cox percolation model, namely stabilization and asymptotically essentially connectedness. (We later show how our model satisfies these two properties.)

Tóbiás [31, 32] studied percolation of a Cox model based on the signal-to-interference. Tóbiás and Jahnel [17] studied signal-to-interference percolation for Cox point processes with random (signal) powers. They showed that percolation was possible in their model provided certain moment conditions on signal powers.

Le Gall, Błaszczyszyn, Cali and En-Najjary wrote papers [21, 22] on device-to-device networks and ultimately used the Cox framework to prove percolation results for a (geometric) Gilbert model [23]. They proved that percolation occurs for sufficiently high user intensities.

In the companion paper [20] of the current one, we adapted the aforementioned Gilbert model of device-to-device networks for connectivity based on signal-to-interference, and gave numerical results supporting the claim that percolation is possible. More so, it was observed numerically that the percolation exhibits a local optimum in terms of the user intensity. For certain parameter regimes, some users are needed for percolation, but, after percolation occurs, too many users will stop percolation.

Other recent work on Cox Poisson-Voronoi percolation includes the paper by Cali, Hinsen, Jahnel and Wary [8] and the paper by Cali, Hinsen, Jahnel [7].

Marchand, Coupier, and Henry [26] recently studied Gilbert-type line-of-sight Cox percolation in a Poisson-Delaunay triangulation, proving that their model also percolates for a certain parameter regime. Naturally, this work is the dual of the aforementioned work based on a Poisson-Voronoi tessellation [23]. The obvious question is whether the methods used in our current work can be applied to study signal-to-interference percolation in such Poisson-Delaunay models.

1.3 Notation and definitions

We refer the reader to the appendix Section C for table listing the notation used throughout this paper. A definition index is located in Section D.

2 Models

The Cox percolation model presented here consists of three main components: a random street layout, transmitting-receiving (device) users and relays randomly positioned on said street layout, and a connectivity model based on signal-to-interference.

2.1 Network model

We describe the network model, which is partially illustrated in Figure 2.

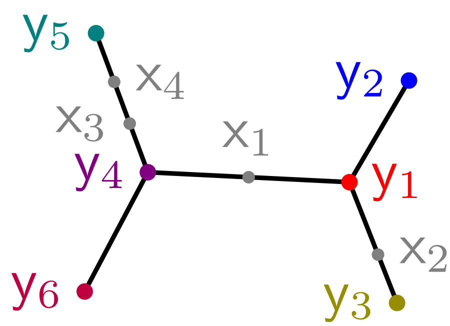

Figure 1: A subsection of a Voronoi street layout S with six relays $\mathsf{y}_1,\ldots,\mathsf{y}_6$ and users $\mathsf{x}_1,\ldots,\mathsf{x}_4$. A relay occupies a crossroads with probability p, while users are located on streets according to homogeneous Poisson point process with (linear) intensity λ. In this example the street system (or graph) has relays $\mathsf{y}_1,\ldots,\mathsf{y}_6$ at all its crossroads (or vertices). We refer to such a five-edge, six-vertex subgraph as a bone; see Definition 5.1. Take the street (or edge) $\mathsf{e}_{1,4}$ that stretches between relays y_1 and y_4. The relay y_4 can receive interference from relays y_1, y_5 and y_6 and from users x_1, x_3, and x_4, as they are within line-of-sight, while all other interference terms are not received. Similarly, relay y_4 can interference from relays y_2, y_3 and y_4, as well as users x_1 and x_2.

2.1.1 Street layout S

We assume a homogeneous Poisson point process Φ with intensity $\mu > 0$ in the plane $\mathbb{R}^2$. We interpret the Poisson-Voronoi tessellation associated with this point process as a street system or layout S. We write $\mathcal{E} := (\mathsf{e}_i)_{i\geq 1}$ and $\mathcal{V} := (\mathsf{v}_i)_{i\geq 1}$ to denote the respective sets of edges and vertices. Of course, in everyday language, the edge set $\mathcal{E}$ and the vertex set $\mathcal{V}$ are respectively the streets and crossroads (or street intersections) in our street layout S. We always consider an edge e as a closed, connected segment of $\mathbb{R}^2$.

2.1.2 Random measure Λ

To scatter users on the street layout S, we first introduce the (random) measure $\Lambda(dx) = \nu_1(S \cap dx)$, where ν_1 denotes the one-dimensional Hausdorff measure on the plane $\mathbb{R}^2$. Owing to the randomness of the street layout S, the measure Λ is a random measure on the plane $\mathbb{R}^2$ with intensity $\gamma := \mathbf{E}[\Lambda([0,1]^2)]$, which is finite and non-null, meaning $0 < \gamma < \infty$. We can interpret the parameter γ as the total edge length per unit volume, which leads to the term *street intensity* for γ. The street intensity is given by $\gamma = 2\sqrt{\mu}$; see the standard reference [9, Section 9.7.2]. For our main percolation results, it is the measure Λ for which we will later require and prove the fact that it possesses two key properties.

2.1.3 Users X^λ

Using the random measure Λ, we can formally scatter users on the street layout S. For $\lambda > 0$, we consider a Cox point process X^λ driven by the random intensity measure $\lambda\Lambda$. The point process X^λ represents the users on the street layout S. In other words, for a given street layout S with edges $\mathcal{E}$, on each edge $\mathsf{e} \in \mathcal{E}$ there exists an independent homogeneous Point process of users with intensity λ, owing to the independence property of the Poisson point process. Hence, the number of users on a given edge $\mathsf{e} \in \mathcal{E}$ is a Poisson random variable with mean $\lambda\nu_1(\mathsf{e})$.

2.1.4 Relays Y

For the relays, we introduce a (doubly stochastic) Bernoulli point process Y on the crossroads $\mathcal{V}$ with parameter p. Conditioned on the measure Λ, the points of Y are placed on the crossroads $\mathcal{V}$ independently with probability p. We refer to the points of Y as (fixed) relays.

2.1.5 Users and relays Z

We assume the users X^λ and relays Y are conditionally independent given their random support Λ. We express this conditional independence as $(X^\lambda \perp\!\!\!\perp Y)|\Lambda$. We write $Z := X^\lambda \cup Y$ to refer to the superposition

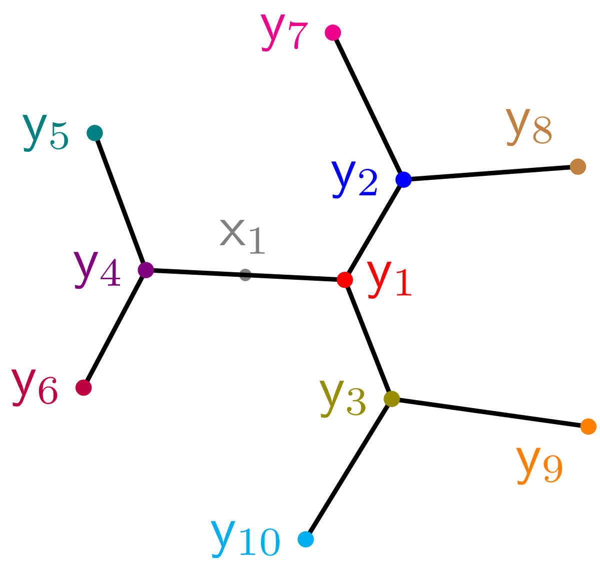

Figure 2: A subsection of a Voronoi street layout S with ten relays $\mathsf{y}_1, \ldots, \mathsf{y}_{10}$ and a single user x_1. When we incorporate interference into the connectivity model, we see that adding a single user x_1 to the street $\mathsf{e}_{1,4}$ may open the street, if it had been previously closed, through multi-hop communication. But then the same user x_1 will also increase the interference experienced by relays y_1 and y_4, which may or may not close some or all of the other streets $\mathsf{e}_{1,3}$, $\mathsf{e}_{1,2}$, $\mathsf{e}_{4,5}$ and $\mathsf{e}_{4,6}$. But the connectivity of all the other streets, such as $\mathsf{e}_{3,9}$ and $\mathsf{e}_{2,8}$, will not be affected by the presence of user x_1.

of users and relays. It is this point process Z for which we want to study percolation under an appropriate connectivity model.

2.2 Connectivity model

2.2.1 Signal-to-(total)-interference model

To introduce the new connectivity model, we first consider a finite collection of transmitter-and-receivers $\{\mathsf{z}_1, \ldots, \mathsf{z}_n\}$ located in $\mathbb{R}^2$. We write $P_{i,j}$ to denote the power of a signal received at z_j originating from a transmitter at z_i. We now introduce the quantity

$$\mathrm{STINR}(\mathsf{z}_i, \mathsf{z}_j) := \frac{P_{i,j}}{N + \eta I_j}\,, \quad \mathsf{z}_i \neq \mathsf{z}_j\,, \tag{1}$$

where $N \geq 0$ is just a noise constant, $\eta \geq 0$ is a technology-dependent parameter, which we can call the *interference factor*, and

$$I_j = \sum_{k \neq j} P_{k,j} \tag{2}$$

is the *total interference*, meaning the sum of signals from the transmitters $\{\mathsf{z}_1, \ldots, \mathsf{z}_n\} \setminus \{\mathsf{z}_j\}$. Typically we will assume that the noise term $N > 0$, which gives meaningful results when there is no interference.

We can call quantity STINR the *signal-to-total-interference-plus-noise ratio*. This ratio is often easier to work with than the related ratio called the *signal-to-interference-plus-noise ratio*, which we denote by SINR. These quantities can be defined in simple terms of each other; see Section A in the appendix. To avoid a mouthful, we will refer to either ratios SINR and STINR as just the signal-to-interference, unless clarification is needed.

2.2.2 Percolation based on signal-to-interference

Under the new connectivity model, we define precisely what we mean by two points of Z being connected.

Definition 2.1 (Connectivity)**.** Two points $\mathsf{z}_i, \mathsf{z}_j \in Z$, where $\mathsf{z}_i \neq \mathsf{z}_j$, are connected provided the following conditions:

1. Line-of-sight: Points z_i and z_j are on the same street, meaning there exists an edge $\mathsf{e} \in \mathcal{E}$ such that $\mathsf{z}_i \in \mathsf{e}$ and $\mathsf{z}_j \in \mathsf{e}$.

2. Sufficient signal-to-total-interference: The respective signal-to-total-interference values of the two points z_i and z_j are larger than some threshold $\tau \in (0, 1/\eta]$ when $\eta > 0$, meaning

$$\text{STINR}(\mathsf{z}_i, \mathsf{z}_j) \geq \tau\,, \quad \text{STINR}(\mathsf{z}_j, \mathsf{z}_i) \geq \tau\,, \quad \mathsf{z}_i \neq \mathsf{z}_j\,, \tag{3}$$

where we recall the definition of STINR given in expression (1), and the total interference is given by

$$I_j = \sum_{\mathsf{z}_k \in Z|_{\mathsf{e}} \setminus \{\mathsf{z}_j\}} P_{k,j}\,, \tag{4}$$

where $Z|_{\mathsf{e}} = Z \cap \mathsf{e}$ denotes the restriction of the point process Z to the street e, and $\text{STINR}(\mathsf{z}_j, \mathsf{z}_i)$ and I_i are defined similarly. (Here I_j is the line-of-sight version of the interference originally defined by expression (2), hence the restricted point process $Z|_{\mathsf{e}}$.)

Definition 2.2 (Connectivity graph $\mathcal{G}$)**.** We denote the connectivity graph formed under the above model as $\mathcal{G}$ or $\mathcal{G}_{p,\lambda,\tau,\eta}$, where the subscripts express the key model variables, excluding the Poisson intensity μ for generating the street layout S.

2.2.3 Multi-hop connectivity

In wireless communications, we can talk about connectivity via a single hop or multiple hops. Under our model, to achieve either form of communication, we need sufficiently large signal-to-interference values in both directions. Clearly single-hop communication is a special case, so we now define multi-hop communication more formally.

Definition 2.3 (Multi-hop connectivity)**.** We say two points z_1 and z_k of the point process Z are multi-hop connected if there exist $k-2$ points of Z , namely $\mathsf{z}_2, ..., \mathsf{z}_{k-1}$, such that for all $i \in 1, ..., k-1$, z_i and z_{i+1} are connected in the sense stated in Definition 2.1.

2.2.4 Path loss model

The standard path loss model assumes the signal power $P_{i,j}$ takes the form

$$P_{i,j} = F_i \ell\left(\|\mathsf{z}_i - \mathsf{z}_j\|\right). \tag{5}$$

Here F_i is a random variable representing propagation phenomena such as fading. In this work, we will set $F_i = 1$ for all $i \geq 1$. (We can always introduce a fading constant $F' > 0$ in the mathematical expressions by replacing all instances of the noise term N with (N/F').) We will assume the path loss function $\ell(t)$ is a non-negative, bounded, continuous and decreasing function in distance $t \geq 0$.

Remark 2.1 (Competing effects)*.* Under the above path loss model, we observe two competing effects that influence the signal-to-interference ratio. For connections to exist:

- Any two points z_i and z_j cannot to be too far away from each other, because the path loss $\ell\left(\|\mathsf{z}_i - \mathsf{z}_j\|\right)$ decreases as the distance $\|\mathsf{z}_i - \mathsf{z}_j\|$ increases.

- Points cannot be too clustered around each other either, because that increases the total interference term I_j due to its summands having the form $(\eta/N)\ell\left(\|\mathsf{z}_k - \mathsf{z}_j\|\right)$.

Bounds on this these two influences, which we can call *separation* and *clustering* effects, will later appear as requirements in Definition 4.7.

For our percolation results, we need to make some assumptions on the path loss model ℓ, which can be compared to those in the seminal work [10, 11] on signal-to-interference percolation.

Definition 2.4 (Path loss model)**.** For our path loss model ℓ, we assume the following properties:

1. $\ell(r)$ is a bounded, continuous, non-negative, decreasing function defined on $[0, \infty)$ such that $\ell(r) \to 0$ as $r \to \infty$;

2. $\ell(r)$ has a well-defined (generalized) inverse;

3. $\ell(0) > [\tau/(1 - \eta\tau)]N$, where we assume the noise constant $N > 0$, while we recall $\tau \in (0, 1/\eta)$ is the signal-to-interference threshold, which is defined in the next section.

The first two path loss properties are physically intuitive. The last property is needed for proving that percolation is possible. Later we will discuss the consequences of relaxing some of the model assumptions.

Remark 2.2 (Relation to the Gilbert model). Under the original model [23], two points $\mathsf{z}_i, \mathsf{z}_j \in Z$, where $\mathsf{z}_i \neq \mathsf{z}_j$, are connected provided the following conditions:

1. Line-of-sight: Points z_i and z_j are on the same street, meaning there exists an edge $\mathsf{e} \in \mathcal{E}$ such that $\mathsf{z}_i \in \mathsf{e}$ and $\mathsf{z}_j \in \mathsf{e}$.

2. Transmission range: Distance between the points z_i and z_j is less than r, meaning $\|\mathsf{z}_i - \mathsf{z}_j\| \leq r$.

This Gilbert-style connectivity requirement results in a connectivity graph, which we denote by $\mathcal{G}^*$ or $\mathcal{G}^*_{p,\lambda,r}$. We can compare the above line-of-sight model [23] to our signal-to-interference model coupled with the standard path loss model described in Section 2.2.4. We rewrite the second condition in terms of the signal-to-total-interference, giving

$$\mathrm{STINR}(\mathsf{z}_i, \mathsf{z}_j) = \frac{(1/N)\ell\left(\|\mathsf{z}_i - \mathsf{z}_j\|\right)}{1 + \eta/N \sum\limits_k \ell\left(\|\mathsf{z}_k - \mathsf{z}_j\|\right)}, \quad \mathsf{z}_i \neq \mathsf{z}_j . \tag{6}$$

Given the connectivity requirement $\mathrm{STINR}(\mathsf{z}_i, \mathsf{z}_j) \geq \tau$, by assuming $N > 0$ and setting $\eta \to 0$, we arrive at the zero interference case $(1/N)\ell\left(\|\mathsf{z}_i - \mathsf{z}_j\|\right) \geq \tau$. Assuming a decreasing and invertible path loss function ℓ, we recover the Gilbert requirement

$$\|\mathsf{z}_i - \mathsf{z}_j\| \leq r, \quad \mathsf{z}_i \neq \mathsf{z}_j , \tag{7}$$

where $r = \ell^{-1}(\tau N)$ and ℓ^{-1} is the inverse of ℓ. Intuitively, any interference in the model reduces connectivity. Consequently, a connectivity graph formed under the signal-to-total-interference requirement is a subgraph of the one formed under the Gilbert requirement, a fact which we will use in the proof of the forthcoming Theorem 2.

2.3 Open streets and crossroads

We now introduce some more definitions for describing percolation of our model.

Definition 2.5 (Open/closed crossroads)**.** We say crossroads $\mathsf{v} \in \mathcal{V}$ is *open* if there exists a relay $\mathsf{y} \in Y$ located at it, which means $Y(\{\mathsf{v}\}) = 1$. We say a crossroads v is *closed* if it is not open.

Definition 2.6 (Open/closed street)**.** We say a street $\mathsf{e} \in \mathcal{E}$ is *open* if a relay is located at both of its crossroads (or street ends) $\mathsf{v}_{\mathsf{e},a}$ and $\mathsf{v}_{\mathsf{e},b}$, and these two relays are connected on the street in the single hop (Definition 2.1) or multi-hop (Definition 2.3) sense. We say a street e is *closed* if it is not open.

3 Main results

Our main results concern graph percolation.

Definition 3.1. We say a graph *percolates* when there exists an infinite component (or cluster) in the graph. The percolation probability is defined as

$$\theta(p, \lambda, \tau, \eta) := \mathbb{P}[\text{Graph } \mathcal{G} \text{ percolates}] , \tag{8}$$

where the probability measure $\mathbb{P} := \mathbb{P}_{p,\lambda,\tau,\eta}$, indicate the dependence of the model parameters.

We can then use this probability to define critical values of parameters. A critical value of the user intensity λ is given by

$$\lambda_{c,1} := \inf_{\lambda} \left[\lambda : \theta(p, \lambda, \tau, \eta) > 0\right], \tag{9}$$

where the other parameters are held fixed. This is just one critical value of the user intensity, but the competing effects described in Remark 2.1 hint at the possibility of there being another critical value for which the graph stops percolating.

Le Gall et al. [23, Theorem 1] proved that the superposition point process Z is ergodic.

Proposition 0.1. *The superposition Z of users X^{λ} and relays Y is mixing and, hence, ergodic.*

Note that the original proof of this result does not rely upon the structure of the Poisson-Voronoi tessellation; see the proof commentary [23, Remark 2] for details. We can apply the above ergodicity result to the translation-invariant event of percolation of the signal-to-interference graph $\mathcal{G}$. The immediate consequence is a zero-one law for the percolation probability θ.

Corollary 1. *For parameters $N > 0$, $\eta \geq 0$, $p \in [0, 1]$, $\lambda \geq 0$, and $\tau \in (0, 1/\eta)$, the percolation probability*

$$\theta(p, \lambda, \tau, \eta) \in \{0, 1\}. \tag{10}$$

In other words, the signal-to-interference graph $\mathcal{G}_{p,\lambda,\tau,\eta}$ percolates almost always or almost never.

We now present the three main results of our percolation work. We start by stating that the signal-to-interference graph $\mathcal{G}_{p,\lambda,\tau,\eta}$ does not percolate when the signal-to-interference threshold τ is too high and the user intensity λ is too low: for noise $N > 0$ and any relay probability $p \in [0, 1]$, the signal-to-interference connectivity graph $\mathcal{G}_{p,\lambda,\tau,\eta}$ does not percolate almost surely for a high enough threshold $\tau \in (0, 1/\eta)$ and low enough user intensity $\lambda > 0$, regardless of the interference factor $\eta \geq 0$.

Theorem 2 (Lower subcritical user intensity)**.** *For any noise value $N > 0$, relay probability $p \in [0, 1]$, and interference factor $\eta \geq 0$, there exists a threshold value $\tau_0 \in (0, 1/\eta)$ and an intensity $\lambda_0 > 0$ (regardless of the value of τ_0) such that if $\tau > \tau_0$ and $\lambda < \lambda_0$, then the percolation probability $\theta(p, \lambda, \tau, \eta) = 0$.*

Proof. We just use the fact that Le Gall et al. [23, Theorem 1] proved that the Gilbert-style graph $\mathcal{G}^*_{p,\lambda,r}$ does not percolate in the equivalent setting. Then our result is immediate because, as observed in Remark 2.2, the signal-to-interference connectivity graph $\mathcal{G}_{p,\lambda,\tau,\eta}$ is a subset of the Gilbert-style graph $\mathcal{G}^*_{p,\lambda,r}$, so

$$\mathcal{G}_{p,\lambda,\tau,0} := \mathcal{G}^*_{p,\lambda,r}, \tag{11}$$

where $r = \ell^{-1}(\tau N)$. In other words, for a threshold $\tau > 0$, the graph $\mathcal{G}_{p,\lambda,\tau,\eta} \subset \mathcal{G}^*_{p,\lambda,r}$, because the interference in the network, due to $\eta \geq 0$, reduces the number of possible edges in the graph $\mathcal{G} = \mathcal{G}_{p,\lambda,\tau,\eta}$. □

The next result says that the signal-to-interference graph $\mathcal{G}$ percolates if the user intensity λ is sufficiently high and the interference factor η is sufficiently low: for a noise value $N > 0$ and a threshold $\tau \in (0, 1/\eta)$, the signal-to-interference connectivity graph $\mathcal{G}$ percolates (with positive probability) for a large enough relay probability $p \in (0, 1)$, a finite user intensity $\lambda < \infty$, and small enough interference factor $\eta > 0$, where the values of p and λ depend on τ but not on each other.

Theorem 3 (Supercritical user intensity)**.** *For any noise value $N > 0$, and threshold $\tau \in (0, 1/\eta)$, there exists an intensity $\lambda_{c,1} = \lambda_{c,1}(\tau) < \infty$, and a probability $p_0 = p_0(\tau) \in (0, 1)$ such that if the relay probability $p \in (p_0, 1]$, and if the user intensity $\lambda > \lambda_{c,1}$, then there exists an interference factor $\eta_0 = \eta_0(\tau, p) > 0$ such that if $\eta < \eta_0(\tau, p)$, then the percolation probability $\theta(p, \lambda, \tau, \eta) > 0$.*

Proof sketch. Theorem 3 uses Lemma 4.3, Lemma 4.4 and Lemma 4.5 found in Section 5.1. □

The final result says that the graph $\mathcal{G}_{p,\lambda,\tau,\eta}$ stops percolating if the user intensity becomes too large: for a noise value $N > 0$ and an interference factor $\eta \geq 0$, and a relay probability $p \in [0, 1]$, there exists a user intensity value $\lambda_{c,2}$ such that if $\lambda > \lambda_{c,2}$ then the signal-to-interference connectivity graph $\mathcal{G}_{r,\lambda,\eta}$ almost surely does not percolate.

Theorem 4 (Upper subcritical user intensity)**.** *For any noise value $N > 0$, interference factor $\eta > 0$, threshold $\tau \in (0, 1/\eta)$ and relay probability $p \in [0, 1]$, there exists an intensity $\lambda_{c,2} = \lambda_{c,2}(\tau) < \infty$ such that if $\lambda > \lambda_{c,2}$, then the percolation probability $\theta(p, \lambda, \tau, \eta) = 0$.*

Proof sketch. Theorem 4 requires Lemma 4.6, Lemma 4.7 and Lemma 4.8, given in Section 5.2, these results being applied to the user intensity $\lambda \to \infty$ for a fixed η. □

4 Proof framework

We present some auxiliary results for our proofs of Theorem 3 and Theorem 4. We start by giving a variation of a classic result in percolation theory.

4.1 Finite dependent site percolation

Le Gall et al. [23] proved the bulk of their results using the result by Liggett, Schonmann and Stacey [24, Theorem 0.0] for a random $\{0, 1\}$-field. We use instead a simpler result presented by Franceschetti and Meester [12, Theorem 2.3.1] whose proof is based on a standard counting argument known as Peierls' argument for showing whether percolation occurs; see, for example, Grimmett [15, pages 16–19]. The same proof idea was used in previous work on signal-to-interference percolation [11, Proposition 3].

Proposition 4.1. *[12, Theorem 2.3.1]*
Consider an infinite square grid G, where sites can be either occupied or empty. Let q be the marginal probability that a given site is occupied. Assuming $k < \infty$, if the random field of site states is k-dependent (Definition 5.3), then there exist probabilities $q_1(k) > 0$ and $q_2(k) < 1$ such that percolation occurs with zero probability, $\theta(q) = 0$, for $q < q_1(k)$ and percolation occurs with positive probability, $\theta(q) > 0$, for $q > q_2(k)$.

We will use Proposition 4.1 in our proofs of Theorem 3 and Theorem 4 found respectively in Section 5.1 and Section 5.2.

4.2 Stabilization and asymptotic essential connectedness

To describe the random streets in the street layout S, we introduced the random measure Λ in Section 2.1.2. For our specific model, we will prove it possesses the random measure properties of *stabilization* and *asymptotic essential connectedness*. These concepts were first introduced by Hirsch, Jahnel, and Cali [16] to study percolation with a focus on spatial dependencies of Cox point processes.

4.2.1 Stabilization

The aforementioned work [16, Definition 2.3] introduced the concept of stabilization of a random measure, which is defined with respect to a stabilization radius. That work proved that the random measure Λ stabilizes with respect to a specific stabilization radius $R^*(x)$, which we present later in Section 4.2.3. But for our work, we will need a new larger radius $R(x)$ to treat our connectivity model, which relies upon seed points surrounding a given Voronoi edge.

4.2.2 Seed points surrounding a Voronoi edge

To define a new stabilization radius $R(x)$, we first introduce some definitions to describe the surrounding structure of an edge. We recall that $\mathcal{E}$ and $\mathcal{V}$ are the respective sets of edges (or streets) and vertices (or crossroads) in the street layout S. Furthermore, the point process Φ on the plane $\mathbb{R}^2$ seeds a Voronoi tessellation. For this seeding point process Φ and a point $x \in \mathbb{R}^2$, we define the distance

$$D_1(x) := \sup\{r > 0 : \Phi(B_x(r)) = 0\}\,, \tag{12}$$

where $B_x(r)$ denotes a ball with radius r centred at the point x. The quantity $D_1(x)$ is often known as the *first contact distance* of the point process Φ.

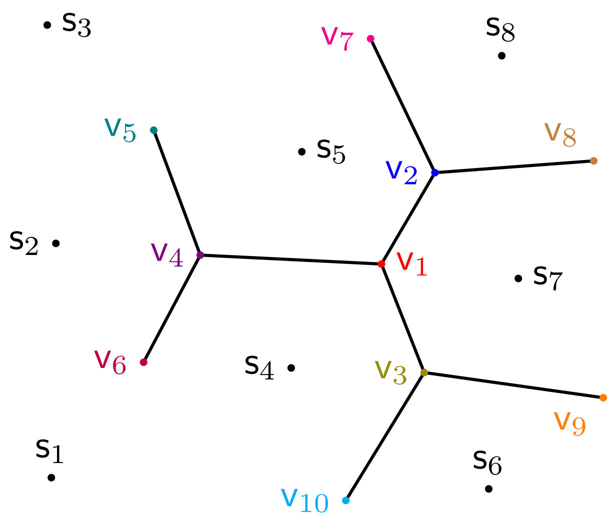

Figure 3: A section of a Voronoi tessellation based on a point process $\{\mathsf{s}_i\}_i$. The corresponding graph has vertices $\{\mathsf{v}_i\}_i$, which are the centres of circumcircles that reveal the points $\{\mathsf{s}_i\}_i$, partially illustrated here in distinct colors. The function ψ_1, defined by expression (13), gives the sets $\psi_1(\mathsf{v}_1) = \{\mathsf{s}_4, \mathsf{s}_5, \mathsf{s}_7\}$, $\psi_1(\mathsf{v}_2) = \{\mathsf{s}_5, \mathsf{s}_7, \mathsf{s}_8\}$ and $\psi_1(\mathsf{v}_4) = \{\mathsf{s}_2, \mathsf{s}_4, \mathsf{s}_5\}$. Defining the edge $\mathsf{e}_{1,2}$ as the edge between vertices v_1 and v_2, the function ψ_2, defined by expression (14), gives the set $\psi_2(\mathsf{e}_{1,4}) = \{\mathsf{s}_1, \mathsf{s}_2, \mathsf{s}_3, \mathsf{s}_4, \mathsf{s}_5, \mathsf{s}_6, \mathsf{s}_7, \mathsf{s}_8\}$.

By the construction of the Poisson-Voronoi street layout S, there are three points (with probability one) of the seeding point process Φ that neighbor a given Voronoi vertex located at x. More specifically, for a vertex located at a point x, we define the triplet of neighboring points

$$\psi_1(x) := \{(\mathsf{y}_1, \mathsf{y}_2, \mathsf{y}_3) \in \Phi : \|x - \mathsf{y}_i\| = D_1(x)\,, i = 1, 2, 3\}\ , \quad x \in \mathcal{V}\,. \tag{13}$$

We now consider the points of the point process Φ that neighbor the two endpoints (or adjacent vertices) that form a given edge in a Voronoi tessellation. More specifically, for a given edge $\mathsf{e} \in \mathcal{E}$, consider the five edges (including edge e) originating from the two endpoints $\mathsf{v}_{\mathsf{e},a}$ and $\mathsf{v}_{\mathsf{e},b}$ (or adjacent vertices) of edge e. These five edges form a figure that we call the *neighborhood* of edge e; for an example, refer to Figure 3. (Our use of the term neighborhood here differs from that used in graph theory.)

We now define the set $\psi_2(\mathsf{e})$ as the set of points of the point process Φ that generate the neighborhood of edge e, namely

$$\psi_2(\mathsf{e}) := \{\mathsf{y} \in \Phi : \exists(\mathsf{y}'_1, \mathsf{y}'_2) \subset \psi_1(\mathsf{v}_{\mathsf{e},i}) : \Phi(B(\mathsf{y}, \mathsf{y}'_1, \mathsf{y}'_2)) = 0\,, i = a, b\,\} \tag{14}$$

where $B(\mathsf{y}, \mathsf{y}'_1, \mathsf{y}'_2)$ is the unique ball defined by its boundary passing through the three points y, y'_1 and y'_2.

In the case of a Poisson-Voronoi tessellation, the number of points in the sets ψ_1 and ψ_2 are both bounded (with probability one). For a single endpoint of an edge $\mathsf{e} \in \mathcal{E}$, such as $\mathsf{v}_{\mathsf{e},a}$, there are (with probability one) at most three ways to choose a pair of points $(\mathsf{y}'_1, \mathsf{y}'_2) \subset \psi_1(\mathsf{v}_{\mathsf{e},a})$ such that the ball $B(\mathsf{y}, \mathsf{y}'_1, \mathsf{y}'_2)$ contains no seed points, meaning $\Phi(B(\mathsf{y}, \mathsf{y}'_1, \mathsf{y}'_2)) = 0$. For an edge $e \in \mathcal{E}$, the set of neighboring points $\psi_2(e)$ contains at most eight distinct seed points (with probability one) of the point process Φ.

4.2.3 Stabilization radius

For $x \in \mathbb{R}^2$, the original work [16, Example 3.1] on the Gilbert-style model used a stabilization radius given by

$$R^*(x) = \inf\{\|x - \mathsf{y}\| : \mathsf{y} \in \Phi\} \tag{15}$$

$$= \sup\{r > 0 : \Phi(B_x(r)) = 0\}\,, \tag{16}$$

and proved that it works for the Cox point process constructed on the Poisson-Voronoi tessellation. When x is not a vertex of the street layout S, so $x \notin \mathcal{V}$, we can still use this stabilization radius.

Otherwise, for the vertex case, our model needs a different choice for the stabilization radius to tackle the additional spatial dependence induced by the interference from the adjacent streets that form part of the neighborhood subgraph. We define that radius by first introducing a distance in relation to a given edge. For an edge $\mathsf{e} \in \mathcal{E}$ and a point $x \in \mathsf{e}$, we define the distance

$$R_2(\mathsf{e}, x) := \sup_{\mathsf{y} \in \psi_2(\mathsf{e})} [\|\mathsf{y} - x\|] \ . \tag{17}$$

Using expressions 15 and 17, we now define our stabilization radius.

Definition 4.1 (Stabilization radius)**.** For all $x \in \mathbb{R}^2$, we define the new stabilization radius as

$$R(x) = \begin{cases} \sup_{\mathsf{e}: x \in \mathsf{e}} [R_2(\mathsf{e}, x)] \,, & \text{if } x \in \mathcal{V}, \\ R^*(x), & \text{otherwise.} \end{cases} \tag{18}$$

Remark 4.1. For $x \in \mathcal{V}$, we have the inequality $R^*(x) \leq R(x)$ for the stabilization radii, which consequently also holds for any $x \in \mathbb{R}^2$.

Definition 4.2 (Stabilization radius of a set)**.** We define the stabilization radius of a set $K \subset \mathbb{R}^2$ such that $R(K) = \sup_{z \in K \cap \mathbb{Q}^2} R(x)$.

4.2.4 Random measure Λ stabilizes with respect to radius $R(x)$

We will now formally state that the random measure Λ stabilizes with respect to the radius $R(x)$. To state our new result, we denote by $Q_a(x) := x + [-a/2, a/2]^2$ the 2-dimensional cube of side a centered at $x \in \mathbb{R}^2$, and we write $Q_a := Q_a(\mathcal{O})$, where $\mathcal{O}$ denotes the origin.

Proposition 4.2. *The random measure Λ on $\mathbb{R}^2$ is* stabilizing *with a random field of stabilization radii $R = \{R(x)\}_{x \in \mathbb{R}^2}$, where the radius $R(x)$ is given in Definition 4.1, and the random field R is defined on the same probability space as Λ and is Λ-measurable and is such that:*

1. *(Λ, R) are jointly stationary;*
2. $\lim_{n \uparrow \infty} \mathbb{P} \left(\sup_{y \in Q_n \cap \mathbb{Q}^2} R(y) < n \right) = 1$*;*
3. *and for all $n \geq 1$, the random variables*

$$\left\{ f\left(\Lambda_{Q_n(x)}\right) \mathbb{1} \left\{ \sup_{y \in Q_n(x) \cap \mathbb{Q}^2} R(y) < n \right\} \right\}_{x \in \varphi} \tag{19}$$

are independent for all bounded measurable functions $f : \mathbf{M} \to [0, +\infty)$ and finite point configurations $\varphi \subset \mathbb{R}^2$ such that for all $x \in \varphi$, $dist_2(x, \varphi \setminus \{x\}) > 3n$, where the distance $dist_2(x, \phi) = \inf \{\|x - \mathsf{y}\| : \mathsf{y} \in \phi\}$ is the Euclidean distance between a point x and collection of points ϕ.

The proof of Proposition 4.2 is given in Section B.

4.2.5 Random measure Λ is asymptotically essentially connected

To give the next new result on the random measure Λ, we write support(Λ) and Λ_{Q_n} denote respectively the measure's support and its restriction to the set Q_n. We now state that the random measure Λ is an asymptotically essentially connected measure as defined by Hirsch, Jahnel, and Cali [16, Definition 2.5].

Proposition 4.3. *The random measure Λ is* asymptotically essentially connected *with respect to the random field $R = \{R(x)\}_{x \in \mathbb{R}^2}$ such that the measure Λ is stabilizing with stabilization radius R, as given in Proposition 4.2, whenever the condition*

$$\sup_{y \in Q_{2n} \cap \mathbb{Q}^2} R(y) < n/2 \tag{20}$$

holds and the conditions:

1. *support*$(\Lambda_{Q_n}) \neq \emptyset$,

2. *support*(Λ_{Q_n}) *is contained in a connected component of support*$(\Lambda_{Q_{2n}})$,

also hold for all $n \geq 1$.

4.2.6 Proof of asymptotic essential connectedness

We assume the stabilization radius $R(x)$ defined in Definition 4.1 for the random measure Λ. We then suppose that $\sup_{x\in\mathbb{Q}_{2n}} R(x) < n/2$. Then $\sup_{x\in\mathbb{Q}_{2n}} R^*(x) < n/2$, and the measure Λ is asymptotically essentially connected, completing the proof. □

4.3 Conditions for open streets and crossroads

We will give conditions for establishing that a street or crossroads are open. But first we introduce some related concepts.

Definition 4.3 (Open/closed street neighborhood)**.** For a street e, we denote by $\mathcal{N}(\mathsf{e})$ the graph composed of the street e and the adjacent streets connected to both its endpoints $\mathsf{v}_{\mathsf{e},a}$ and $\mathsf{v}_{\mathsf{e},b}$. We call the graph $\mathcal{N}(\mathsf{e})$ a *street neighborhood*. We say that a street neighborhood $\mathcal{N}(\mathsf{e})$ is open if all of its streets $\mathcal{N}(\mathsf{e})$ are open in the sense of Definition 2.6. We say that a street neighborhood $\mathcal{N}(\mathsf{e})$ is *closed* if it is not open.

Definition 4.4 (Open/closed street segment)**.** For a street $\mathsf{e} \in \mathcal{E}$, let s be (in the topologically sense) a closed and connected segment of the street e. Let $\mathsf{z}_1, \dots, \mathsf{z}_n$ be all the points of Z belonging to s (if any exist). In the sense of percolation, we say the segment s is *open* if and only if $n \geq 2$ and the points $\mathsf{z}_1, \dots, \mathsf{z}_n$ are multi-hop connected, when considering the interference from only the points $\mathsf{z}_1, \dots, \mathsf{z}_n$. The segment s is *closed* if it is not open.

Observe that any segment is also open if it has at least two points and is a part of an open edge.

Definition 4.5 (Neighboring points on the same street)**.** For any two distinct points $\mathsf{z}_i, \mathsf{z}_j \in Z$ existing on the same street e, we write $\mathsf{z}_i \sim \mathsf{z}_j$ to denote they are neighbors. In other words, for two points $\mathsf{z}_i, \mathsf{z}_j \in Z \cap \mathsf{e}$, where $\mathsf{z}_i \neq \mathsf{z}_j$, the notation $\mathsf{z}_i \sim \mathsf{z}_j$ indicates that the interval $(\mathsf{z}_i, \mathsf{z}_j)$ contains no points of Z, so $(\mathsf{z}_i, \mathsf{z}_j) \cap Z = \emptyset$.

Definition 4.6. For a given street e, we write $d_{\max}(\mathsf{e})$ and $d_{\min}(\mathcal{N}(\mathsf{e}))$ to denote the largest distance and smallest distance between any two neighboring points of Z existing respectively on street e and its neighborhood $\mathcal{N}(\mathsf{e})$. More precisely, we define the distances as

$$d_{\max}(\mathsf{e}) = \max_{\substack{\mathsf{z}_i \sim \mathsf{z}_j \\ \mathsf{z}_i, \mathsf{z}_j \in Z \cap (\mathsf{e})}} (|\mathsf{z}_i - \mathsf{z}_j|) \tag{21}$$

and

$$d_{\min}(\mathcal{N}(\mathsf{e})) = \min_{\substack{\mathsf{z}_i \sim \mathsf{z}_j \\ \mathsf{z}_i, \mathsf{z}_j \in Z \cap \mathcal{N}(\mathsf{e})}} (|\mathsf{z}_i - \mathsf{z}_j|) \tag{22}$$

We now define two conditions on the distances $d_{\max}(\mathsf{e})$ and $d_{\min}(\mathcal{N}(\mathsf{e}))$, which reflect the two competing effects described in Remark 2.1.

Definition 4.7. For a noise constant $N > 0$, a threshold $\tau \in (0, 1/\eta)$ and a given $\delta > 0$, we define two conditions on the distances between points of Z on a given street $\mathsf{e} \in \mathcal{E}$ as the following:

Condition $A_\delta(\mathsf{e})$:

$$(1/N)\ell(d_{\max}(\mathsf{e})) \geq \tau + \delta\,. \tag{23}$$

Condition $B_\delta(\mathcal{N}(\mathsf{e}))$**:**

$$(\eta/N) \sum_{X \in X^\lambda|_{\mathcal{N}(\mathsf{e})}} \ell(d_{\min}(\mathcal{N}(\mathsf{e}))) \leq \frac{\delta}{\tau}\,. \tag{24}$$

Condition $A_\delta(\mathsf{e})$ bounds the degree of the separation of points on a given street e, while Condition $B_\delta(\mathcal{N}(\mathsf{e}))$ bounds the degree of clustering of points in the neighborhood $\mathcal{N}(\mathsf{e})$. That completes our onslaught of definitions.

We now present a result involving the distance conditions defined in Definition 4.7.

Lemma 4.1. *Let* e *be a street with endpoints* $\mathsf{v}_{\mathsf{e},a}$ *and* $\mathsf{v}_{\mathsf{e},b}$*. If there are relays at endpoints* $\mathsf{v}_{\mathsf{e},a}$ *and* $\mathsf{v}_{\mathsf{e},b}$*, meaning* $\mathsf{v}_{\mathsf{e},a}, \mathsf{v}_{\mathsf{e},b} \in Y$*, and there exists a* $\delta > 0$ *such that Condition* $A_\delta(\mathsf{e})$ *and Condition* $B_\delta(\mathcal{N}(\mathsf{e}))$ *are met, then street* e *is open.*

Proof. Assume a finite collection of points $\mathsf{z}_1, \ldots, \mathsf{z}_n$ ordered on the street e in either direction, where (arbitrarily) $\mathsf{z}_1 = \mathsf{v}_{\mathsf{e},a}$ and $\mathsf{z}_n = \mathsf{v}_{\mathsf{e},b}$. We consider the case of a user $\mathsf{z}_{n-1} \in \mathsf{e}$ trying to connect to the neighboring $\mathsf{z}_n \in \mathsf{e}$, which is a relay. The signal-to-total-interference is bounded by

$$\text{STINR}(\mathsf{z}_{n-1}, \mathsf{z}_n) \geq \frac{(1/N)\ell(d_{n-1,n})}{1 + (\eta/N) \sum\limits_{\mathsf{z}_k \in \mathcal{N}(\mathsf{e}) \setminus \mathsf{z}_n} \ell(d_{k,n})}\,, \tag{25}$$

where $d_{k,n} = \|\mathsf{z}_k - \mathsf{z}_n\|$ is the (path) distance. We naturally obtain an inequality if we replace the distance $d_{n-1,n}$ with $d_{\max}(\mathsf{e})$ in the numerator on the right-hand side, which allows us to use inequality (23) of condition $A_\delta(\mathsf{e})$, giving

$$\text{STINR}(\mathsf{z}_{n-1}, \mathsf{z}_n) \geq \frac{(\tau + \delta)}{1 + (\eta/N) \sum\limits_{\mathsf{z}_k \in \mathcal{N}(\mathsf{e}) \setminus \mathsf{z}_n} \ell(d_{k,n})}\,. \tag{26}$$

Similarly, in the denominator, replacing the (path) distances with the distance $d_{\min}(\mathcal{N}(\mathsf{e}))$ results in the largest possible total interference value. Then using inequality (24) of condition $B_\delta(\mathcal{N}(\mathsf{e}))$ gives

$$\text{STINR}(\mathsf{z}_{n-1}, \mathsf{z}_n) \geq \frac{(\tau + \delta)}{1 + \delta/\tau} = \tau\,. \tag{27}$$

and, hence, point z_{n-1} is connected to the relay z_n. This result is also true when the receiver is a user, meaning the z_{j-1}-to-z_j case, where $j = 2, \ldots, n-1$. The proof remains the same except in the denominator we sum only the interference terms over the single street e, recalling that only interference from the same street matters because of the line-of-sight requirement. Hence, the result is true for all points z_j on the street e, so the street is open, completing the proof. ☐

Lemma 4.2. *Let* e *be a street such that there are relays at the endpoints* $\mathsf{v}_{\mathsf{e},a}$ *and* $\mathsf{v}_{\mathsf{e},b}$*, meaning* $\mathsf{v}_{\mathsf{e},a}, \mathsf{v}_{\mathsf{e},b} \in Y$*. For any given threshold* $\tau > 0$*, there exists the limit*

$$\lim_{\lambda \to \infty} \lim_{\eta \to 0} [\mathbb{P}(\mathsf{e} \textit{ is open })|\Lambda] = 1\,. \tag{28}$$

Proof. Lemma 4.1 implies that, for a given $\delta > 0$ and street e, we only have to prove the limit

$$\lim_{\lambda \to \infty} \lim_{\eta \to 0} [\mathbb{P}(A_\delta \cap B_\delta)|\Lambda] = 1\,. \tag{29}$$

Due to Condition 3 in Definition 2.4, we have the inequality $\ell(0) > N\tau/(1 - \eta\tau) \geq N\tau$. Assume $\delta > 0$ such that $\ell(0) > N(\tau + \delta)$. Further assume $\varepsilon > 0$. By increasing the user density λ on the street e, we argue that, for large enough user intensity $\lambda > 0$, we obtain the probability bound

$$\mathbb{P}(A_\delta) \geq 1 - \varepsilon/2\,. \tag{30}$$

To see this explicitly, note that for a Poisson point process with intensity λ on edge e of length $\ell_1(\mathsf{e})$, the maximum gap $d_{\max}(\mathsf{e})$ between consecutive points satisfies $\mathbb{E}[d_{\max}(\mathsf{e})] \leq C/\lambda$ for some constant C depending on $\ell_1(\mathsf{e})$. By Markov's inequality, for any $t > 0$:

$$\mathbb{P}(d_{\max}(\mathsf{e}) > t) \leq \frac{\mathbb{E}[d_{\max}(\mathsf{e})]}{t} \leq \frac{C}{\lambda t}\,. \tag{31}$$

Choosing $t = \ell^{-1}(N(\tau+\delta))$, we require $(1/N)\ell(t) \geq \tau + \delta$, so:

$$\mathbb{P}(A_\delta) = \mathbb{P}\left((1/N)\ell(d_{\max}) \geq \tau+\delta\right) \tag{32}$$
$$= \mathbb{P}\left(d_{\max} \leq \ell^{-1}(N(\tau+\delta))\right) \tag{33}$$
$$\geq 1 - \frac{C}{\lambda \ell^{-1}(N(\tau+\delta))}\,. \tag{34}$$

Hence, choosing $\lambda > \frac{2C}{\varepsilon \ell^{-1}(N(\tau+\delta))}$ gives $\mathbb{P}(A_\delta) \geq 1 - \varepsilon/2$.

The local finiteness of the users on the streets implies that

$$\mathbb{P}[Z(\mathcal{N}(\mathsf{e})) < \infty] = 1]\,. \tag{35}$$

Therefore, by the path loss ℓ being bounded and assuming $\eta > 0$, a small enough interference factor η gives the probability bound

$$\mathbb{P}\left[Z(\mathcal{N}(\mathsf{e}))\ell(d_{\min}(\mathcal{N}(\mathsf{e}))) < \frac{N}{\eta}\frac{\delta}{\tau}\right] \geq 1 - \varepsilon/2\,. \tag{36}$$

Furthermore, combining the probability bounds $\mathbb{P}(A_\delta) \geq 1-\varepsilon/2$ and $\mathbb{P}(B_\delta) \geq 1 - \varepsilon/2$ gives the bound

$$\mathbb{P}(A_\delta \cap B_\delta\ ) = \mathbb{P}(A_\delta) - \mathbb{P}(A_\delta \setminus B_\delta) \tag{37}$$
$$\geq \mathbb{P}(A_\delta) - \mathbb{P}(B_\delta^c) = 1-\varepsilon, \tag{38}$$

where B_δ^c denotes the complement of the event B_δ, which completes the proof. □

5 Proofs of Theorem 3 and Theorem 4

5.1 Proof of Theorem 3

For this proof, we will make use of a specific subgraph, which we call a *bone*; see Figure 1 or 4.

Definition 5.1 (Bone)**.** For an open edge $\mathsf{e} \in \mathcal{E}$ with endpoints v_1 and v_2, where both v_1 and v_2 have degree 3, we define the *bone* of e as the subgraph consisting of e together with all edges adjacent to v_1 and v_2.

We now define a percolation model on the integer lattice $\mathbb{Z}^2$, whose percolation will indicate the percolation of the connectivity graph $\mathcal{G}$.

Definition 5.2. For $n \geq 1$, say a site $z \in \mathbb{Z}^2$ is n-good if the following conditions are satisfied:

A. The stabilization radius $R(x)$ is bounded such that $\max_{x \in Q_{9n}(nz)} R(x) < 9n$.

B. There exists a street neighborhood $\mathcal{N}(\mathsf{e})$ entirely contained in $Q_n(nz)$. We denote this by $\mathcal{N}(\mathcal{E}) \cap Q_n(nz) \neq \emptyset$.

C. Among the street neighborhoods that are located completely within the box $Q_n(nz)$, if any, there exists at least one street neighborhood $\mathcal{N}(\mathsf{e})$ that is open in the sense of Definition 4.3.

D. All crossroads inside the box $Q_{6n}(nz)$ are open in the sense of Definition 2.5.

E. Every two open edge neighborhoods $\mathcal{N}(\mathsf{e}), \mathcal{N}(\mathsf{e}') \in \mathcal{N}(\mathcal{E}) \cap Q_{3n}(nz)$ are connected by a path of open edges in the graph $\mathcal{G} \cap Q_{6n}(nz)$, and for each vertex of this path, all the adjacent edges are completely contained within the box $Q_{9n}(nz)$; see Figure 4.

We say a site $z \in \mathbb{Z}^2$ is n-bad if it is not n-good.

We have defined the n-good sites above so as to satisfy the next result.

Lemma 4.3. *Percolation of the process of n-good sites implies percolation of the signal-to-interference connectivity graph $\mathcal{G}$.*

Proof. Let $\mathcal{C}$ be an infinite connected component of n-good sites. Consider the points $z, z' \in \mathcal{C}$ such that $\|z - z'\|_1 = 1$. Without loss of generality, assume $z = (a_1, a_2)$ for some $a_1, a_2 \in \mathbb{Z}$ and $z' = (a_1 + 1, a_2)$. By condition C. in the definition of n-goodness, we can find open edge neighborhoods $\mathcal{N}(\mathsf{e}) \in \mathcal{N}(\mathcal{E}) \cap Q_n(nz)$ and $\mathcal{N}(\mathsf{e}') \in \mathcal{N}(\mathcal{E}) \cap Q_n(nz')$. Since

$$Q_n(nz) = [na_1 - n/2, na_1 + n/2] \times [na_2 - n/2, na_2 + n/2]\,, \tag{39}$$
$$Q_n(nz') = [na_1 + n/2, na_1 + 3n/2] \times [na_2 - n/2, na_2 + n/2]\,, \tag{40}$$
$$Q_{3n}(nz) = [na_1 - 3n/2, na_1 + 3n/2] \times [na_2 - 3n/2, na_2 + 3n/2]\,, \tag{41}$$
$$Q_{6n}(nz) = [na_1 - 3n, na_1 + 3n] \times [na_2 - 3n, na_2 + 3n]\,, \tag{42}$$

we have $Q_n(nz') \subset Q_{3n}(nz)$ and so $\mathcal{N}(\mathsf{e}') \in \mathcal{N}(\mathcal{E}) \cap Q_n(nz')$ implies $\mathcal{N}(\mathsf{e}') \in \mathcal{N}(\mathcal{E}) \cap Q_{3n}(nz)$. Furthermore, using the fact that $\mathcal{N}(\mathsf{e}) \in \mathcal{N}(\mathcal{E}) \cap Q_n(nz) \subset \mathcal{N}(\mathcal{E}) \cap Q_{3n}(nz)$ and the edge neighborhoods $\mathcal{N}(\mathsf{e})$ and $\mathcal{N}(\mathsf{e}')$ are both open, then condition E. in the definition of n-goodness implies that edge neighborhoods $\mathcal{N}(\mathsf{e})$ and $\mathcal{N}(\mathsf{e}')$ are connected by a path $\mathcal{L}$ (of open edges) in the graph $\mathcal{G} \cap Q_{6n}(nz)$. Therefore, the path $\mathcal{L}$ also connects edge e and edge e' in the graph $\mathcal{G}$, giving rise to an infinite connected component in the graph $\mathcal{G}$. This concludes the proof of Lemma 4.3. □

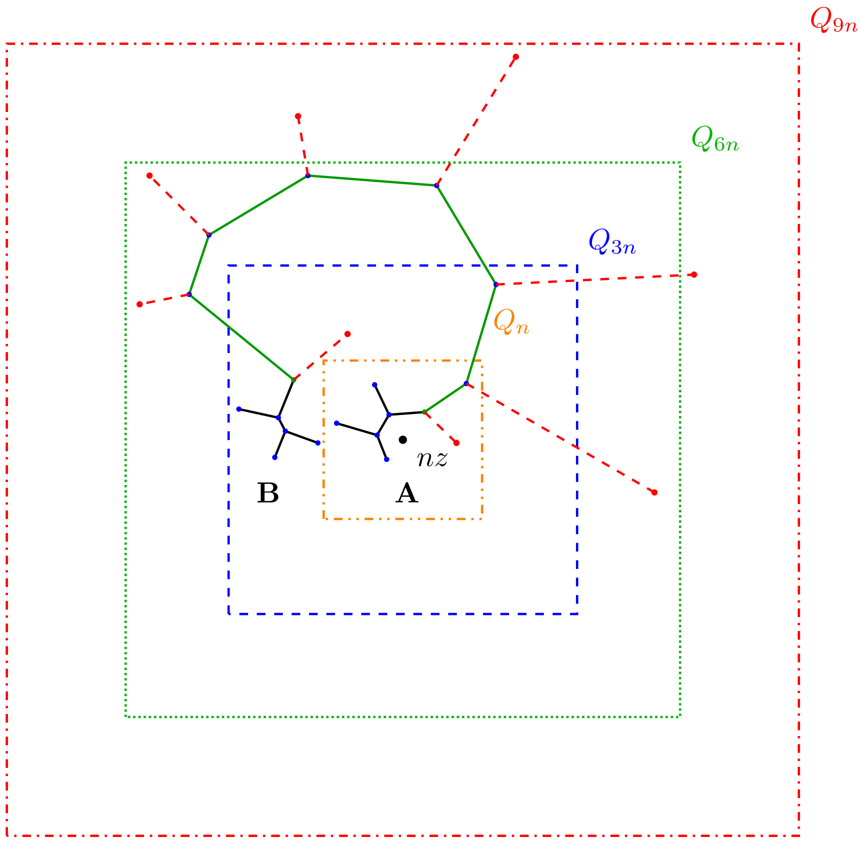

Figure 4: Conceptual diagram illustrating a key part of our proof for Theorem 3, featuring four boxes, which are not drawn to scale, all centered at the lattice site nz. **Box** Q_n completely contains the subgraph **Bone** A, which consists of open edges; see Definition 5.1. **Box** Q_{3n} completely contains the two separate bone structures, **Bone** A and **Bone** B, which are are not connected by any open path contained within that box. The two bones are, however, connected by an open path (shown in green) that remains entirely within **Box** Q_{6n}. Any edges (shown in red), which are not necessarily open, connected to this connecting open path must terminate within **Box** Q_{9n} and not extend beyond it.

Before presenting the next result, we need to define k-dependence.

Definition 5.3. Let $\mathbf{X} = (X_z)_{z \in \mathbb{Z}^2}$ be a discrete random field. Let $k \geq 1$. Then $\mathbf{X}$ is said to be k-dependent if for all $q \geq 1$ and all $\{s_1, \ldots s_q\} \subset \mathbb{Z}^2$ are finite with the property that for all $i \neq j$, we have $\|s_i - s_j\|_\infty > k$, such that the random variables $(X_{s_i})_{1 \leq i \leq q}$ are independent.

Lemma 4.4. *For $z \in \mathbb{Z}^2$, set $\xi_z := \mathbb{1}\{z \text{ is } n\text{-good}\}$. Then the collection of random variables $(\xi_z)_{z\in\mathbb{Z}^2}$ is an 39-dependent random field.*

Proof. It is enough to prove that for all finite $\psi = \{z_1, \dots z_q\} \subset \mathbb{Z}^2$ such that for all $i \neq j$ and $\|z_i - z_j\|_\infty > 39$, we have

$$\mathbf{E}\left(\prod_{i=1}^{q} \xi_{z_i}\right) = \prod_{i=1}^{q} \mathbf{E}(\xi_{z_i})\,. \tag{43}$$

Recalling n-goodness defined in Definition 5.2, we denote respectively by A_z, B_z, C_z, D_z, and E_z the events that the n-goodness conditions A.,B., C., D., and E. all hold for $z \in \mathbb{Z}^2$. Hence, for all $z \in \mathbb{Z}^2$, we have the bound

$$\xi_z = \mathbb{1}\{A_z\}\mathbb{1}\{B_z\}\mathbb{1}\{C_z\}\mathbb{1}\{D_z\}\mathbb{1}\{E_z\}\,. \tag{44}$$

Note whenever $z \in \mathbb{Z}^2$, the indicators $\mathbb{1}\{A_z\}$ and $\mathbb{1}\{B_z\}$ are Λ-measurable, so we can write

$$\mathbf{E}\left(\prod_{i=1}^{q} \xi_{z_i}\right) = \mathbf{E}\left[\mathbf{E}\left(\prod_{i=1}^{q} \xi_{z_i}\middle|\Lambda\right)\right] \tag{45}$$

$$= \mathbf{E}\left[\prod_{i=1}^{q} \mathbb{1}\{A_{z_i}\}\mathbb{1}\{B_{z_i}\}\mathbf{E}\left(\prod_{i=1}^{q} \mathbb{1}\{C_{z_i} \cap D_{z_i} \cap E_{z_i}\}\middle|\Lambda\right)\right]. \tag{46}$$

Now note that conditioned on Λ, for each $1 \leq i \leq q$, the event $C_{z_i} \cap D_{z_i} \cap E_{z_i}$ only depends on the configuration of users X^λ and relays Y inside the 2-dimensional cube $Q_{9n}(nz_i)$. Since the set of points ψ satisfies the condition for all $i \neq j$, the inter-site distances $\|z_i - z_j\|_\infty > 39$, then for all $i \neq j$ we have $\|nz_i - nz_j\|_\infty > 39n$. In particular, the cubes $\{Q_{9n}(nz_i)) : 1 \leq i \leq q\}$ are disjoint, so for all $i \neq j$, we have the empty intersection

$$Q_{9n}(nz_i) \cap Q_{9n}(nz_j) = \emptyset\,. \tag{47}$$

Given Λ, recall that X^λ has the distribution of a Poisson point process and Y has the distribution of a Bernoulli point process. Then by the complete independence of Poisson and Bernoulli processes, we have

$$\mathbf{E}\left(\prod_{i=1}^{q} \xi_{z_i}\right) = \mathbf{E}\left[\prod_{i=1}^{q} \mathbb{1}\{A_{z_i}\}\mathbb{1}\{B_{z_i}\}\mathbf{E}\left(\prod_{i=1}^{q} \mathbb{1}\{C_{z_i} \cap D_{z_i} \cap E_{z_i}\}\middle|\Lambda\right)\right] \tag{48}$$

$$= \mathbf{E}\left[\prod_{i=1}^{q} \mathbb{1}\{A_{z_i}\}\mathbb{1}\{B_{z_i}\}\prod_{i=1}^{q} \mathbf{E}\left(\mathbb{1}\{C_{z_i} \cap D_{z_i} \cap E_{z_i}\}\middle|\Lambda\right)\right] \tag{49}$$

$$= \mathbf{E}\left[\prod_{i=1}^{q} \mathbb{1}\{A_{z_i}\}\prod_{i=1}^{q} \mathbf{E}\left(\mathbb{1}\{B_{z_i} \cap C_{z_i} \cap D_{z_i} \cap E_{z_i}\}\middle|\Lambda\right)\right] \tag{50}$$

$$= \mathbf{E}\left[\prod_{i=1}^{q} \mathbb{1}\{R(Q_{9n}(nz_i)) < 9n\}f(\Lambda_{Q_{9n}(nz_i)})\right], \tag{51}$$

where the function

$$f(\Lambda_{Q_{9n}(x)}) := \mathbf{E}\left(\mathbb{1}\{B_x \cap C_x \cap D_x \cap E_x\} \mid \Lambda\right) \tag{52}$$

is a bounded measurable deterministic function of $\Lambda_{Q_{9n}(x)}$.

Now the set $\varphi := \{nz_1, \dots, nz_p\} \subset \mathbb{R}^2$ is finite and, for all $i \neq j$, it satisfies the condition

$$\|nz_i - nz_j\|_\infty > 39n > 27\sqrt{2}n. \tag{53}$$

For all $i \neq j$, we have the inter-site distance condition $\|nz_i - nz_j\| > 27n$. Hence, for all $x \in \varphi$, the set φ satisfies the condition

$$\mathrm{dist}_2(x, \varphi \setminus \{x\}) > 27n = 3 \times 9n. \tag{54}$$

Replacing n with $9n$, we can now apply Property 3 in Proposition 4.2 regarding stabilization so that the random variables appearing on the right-hand side of expression (51) are all independent. Hence, we arrive at

$$\mathbf{E}\left(\prod_{i=1}^{q}\xi_{z_i}\right) = \prod_{i=1}^{q}\mathbf{E}\left[\mathbb{1}\{R(Q_{9n}(nz_i) < 9n\}f(\Lambda_{Q_{9n}(nz_i)})\right] \tag{55}$$

$$= \prod_{i=1}^{q}\mathbf{E}(\xi_{z_i}), \tag{56}$$

which concludes the proof of Lemma 4.4. □

We consider an arbitrary site, which we can choose to be the origin $\mathcal{O} \in \mathbb{Z}^2$ due to stationarity. For finite n, we now prove that we can make arbitrarily close to one the probability of this arbitrary site being n-good, which we achieve by first taking a large enough n and then a high enough user intensity λ. We now formally state this result.

Lemma 4.5. *For any threshold $\tau > 0$ we have*

$$\lim_{n\uparrow\infty}\lim_{\substack{p\uparrow 1,\\ \lambda\uparrow\infty}}\lim_{\eta\downarrow 0}\mathbb{P}(\mathcal{O}\textit{ is } n\textit{-good}) = 1. \tag{57}$$

Proof. We will prove the complementary limit

$$\lim_{n\uparrow\infty}\lim_{\substack{p\uparrow 1,\\ \lambda\uparrow\infty}}\lim_{\eta\downarrow 0}\mathbb{P}(\mathcal{O}\text{ is } n\text{-bad}) = 0. \tag{58}$$

Take any $\epsilon > 0$. Recalling n-goodness described in Definition 5.2, denote respectively by A,B,C,D, and E the events that the n-goodness conditions A., B., C., D., and E. hold for $z = \mathcal{O}$. Denote by $\tilde{A}$ the event that $R(Q_{9n}) < n/2$.

Note that $\tilde{A} \subset A$, which gives

$$\mathbb{P}(\mathcal{O}\text{ is } n\text{-bad}) = \mathbb{P}(A^c \cup B^c \cup C^c \cup D^c \cup E^c) \tag{59}$$

$$\leq \mathbb{P}(\tilde{A}^c \cup B^c \cup C^c \cup D^c \cup E^c) \tag{60}$$

$$\leq \mathbb{P}(\tilde{A}^c) + \mathbb{P}(B^c) + \mathbb{P}(B \cap C^c \cap D) + \mathbb{P}(D^c) + \mathbb{P}(\tilde{A} \cap D \cap E^c). \tag{61}$$

By partitioning the cube Q_{9n} into 18^2 subcubes $(K_i)_{1\leq i\leq 18^2}$ of side length $n/2$, we obtain

$$\mathbb{P}(\tilde{A}^c) = \mathbb{P}(R(Q_{9n}) \geq n/2) \tag{62}$$

$$= \mathbb{P}\left(\bigcup_{i=1}^{18^2}\{R(K_i) \geq n/2\}\right) \tag{63}$$

$$\leq 18^2\,\mathbb{P}(R(Q_{n/2}) \geq n/2), \tag{64}$$

where the last line is due to the stationarity of the R variables. Given Property 2 in Proposition 4.2, we arrive at the limit $\lim_{n\uparrow\infty}\mathbb{P}(\tilde{A}^c) = 0$.

Furthermore, by increasing the window size n, the window will eventually contain a complete neighborhood, consequently obtaining the limit $\lim_{n\uparrow\infty}\mathbb{P}(B^c) = 0$. We now fix n large enough such that $\mathbb{P}(\tilde{A}^c) \leq \epsilon/5$ and $\mathbb{P}(B^c) \leq \epsilon/5$. To treat the probability $\mathbb{P}(B \cap C^c \cap D)$, we re-write it as

$$\mathbb{P}(C^c \cap B \cap D) = [1 - \mathbb{P}(C|B \cap D)]\mathbb{P}(B \cap D) \tag{65}$$

Lemma 4.2 implies the limit

$$\lim_{\lambda\uparrow\infty}\lim_{\eta\downarrow 0}\mathbb{P}(C|B \cap D) = 1 \tag{66}$$

Hence, for large enough $\lambda < \infty$ (depending on n) and small enough η (depending on n and λ) we have $\mathbb{P}(B \cap C^c \cap D) \leq \epsilon/5$.

Similarly, the probability

$$\mathbb{P}(D^c) = \mathbb{P}(\exists v \in \mathcal{V} \cap Q_{6n} : \, v \text{ is closed}) \tag{67}$$

also converges to zero when $p \uparrow 1$ (for fixed n). Hence, for large enough $p < 1$ (depending on n), we have the bound $\mathbb{P}(D^c) \leq \epsilon/5$.

For the final part of the proof, we will prove the limit

$$\lim_{\lambda\to\infty} \lim_{\eta\to 0} \mathbb{P}(\tilde{A} \cap D \cap E^c) = 0, \tag{68}$$

which we do by first writing the probability

$$\mathbb{P}(E^c \cap \tilde{A} \cap D) = [1 - \mathbb{P}(E|\tilde{A} \cap D)]\mathbb{P}(\tilde{A} \cap D)\,. \tag{69}$$

Hence, we need to prove the limit

$$\lim_{\lambda\to\infty} \lim_{\eta\to 0} \mathbb{P}(E|\tilde{A} \cap D) = 1. \tag{70}$$

To prove this, we will first use two purely geometric arguments, and then one probabilistic argument. First, given the event $\tilde{A}$, there exists a point $y \in Q_{6n}(nz)$ such that $R(y) < n/2$. Then, owing to the result on asymptotic essential connectedness stated in Proposition 4.3, we have the non-empty support $\text{supp}(\Lambda_{Q_{3n}}) \neq \emptyset$, and there also exists a connected component $\mathcal{C}$ of the set $\text{supp}(\Lambda_{Q_{6n}})$ such that the set $\text{supp}(\Lambda_{Q_{3n}}) \subset \mathcal{C} \subset \text{supp}(\Lambda_{Q_{6n}})$.

Second, we claim all edges e adjacent to the component $\mathcal{C}$ are completely contained within the box $Q_{9n}(nz)$. To prove this claim, we suppose the converse such that there exists an edge e with one endpoint $\mathsf{v}_{\mathsf{e},a} \in Q_{6n}(nz)$, while the other endpoint $\mathsf{v}_{\mathsf{e},b} \notin Q_{9n}(nz)$. Given the edge e extends beyond the box $Q_{9n}(nz)$, there is a point of intersection $u \in \mathsf{e} \cap \delta Q_{9n}(nz)$, where $\delta Q_{9n}(nz)$ is the boundary of the box $Q_{9n}(nz)$. But then, the Voronoi construction means there exists a point s of the underlying (Poisson) point process Φ such that the distances $d(\mathsf{v}_{\mathsf{e},a}, \mathsf{s}) < n/2$ and $d(u, \mathsf{s}) < n/2$, so the distance $d(\mathsf{v}_{\mathsf{e},a}, u) < n$, contradicting our initial assumption that $\mathsf{v}_{\mathsf{e},a} \in Q_{6n}(nz)$ and $\mathsf{v}_{\mathsf{e},b} \notin Q_{9n}(nz)$, thus proving our claim.

Finally, the above two geometric arguments allow us to make the concluding probabilistic argument.By Lemma 4.2, we argue that for any edge $\mathsf{e} \in \mathcal{E} \cap Q_{6n}$ there is the limit

$$\lim_{\lambda\to\infty} \lim_{\eta\to 0} [\mathbb{P}(\mathsf{e} \text{ is open })|D] = 1\,. \tag{71}$$

Since the box Q_{6n} contains finitely many edges almost surely, we can apply a union bound over all edges. Specifically, conditional on Λ and event D, let $K_n = |\mathcal{E} \cap Q_{6n}|$ denote the number of edges in Q_{6n}. For sufficiently large λ and small η, Lemma 4.2 ensures

$$\mathbb{P}(\mathsf{e} \text{ is open}|\Lambda, D) \geq 1 - \frac{\epsilon}{5K_n} \tag{72}$$

for each edge $\mathsf{e} \in \mathcal{E} \cap Q_{6n}$. The union bound then gives:

$$\begin{aligned} \mathbb{P}(\text{all edges in } Q_{6n} \text{ are open}|\Lambda, D) &\geq 1 - \sum_{\mathsf{e}\in\mathcal{E}\cap Q_{6n}} \mathbb{P}(\mathsf{e} \text{ is closed}|\Lambda, D) && (73) \\ &\geq 1 - K_n \cdot \frac{\epsilon}{5K_n} && (74) \\ &= 1 - \frac{\epsilon}{5}\,, && (75) \end{aligned}$$

which proves the limit in expression (70). Consequently, we can find a user intensity $\lambda < \infty$ large enough (depending on n) such that the probability $\mathbb{P}(\tilde{A} \cap D \cap E^c) \leq \epsilon/5$. We recall that $\epsilon > 0$ was arbitrary, so the edge e is open, which completes the proof of Lemma 4.5. □

We now consider occupied sites as the n-good sites. Due to Lemma 4.4, we can now use Proposition 4.1 with $k = 39$. Lemma 4.5 then proves we can choose n, λ and p large enough, there exists η_0 such that if $\eta < \eta_0$, $q > q_2$, making the process of n-good sites percolate. Consequently, by Lemma 4.3, the connectivity graph $\mathcal{G}_{p,\lambda,\tau,\eta}$ with these values of λ and p percolates, which completes the proof of Theorem 3.

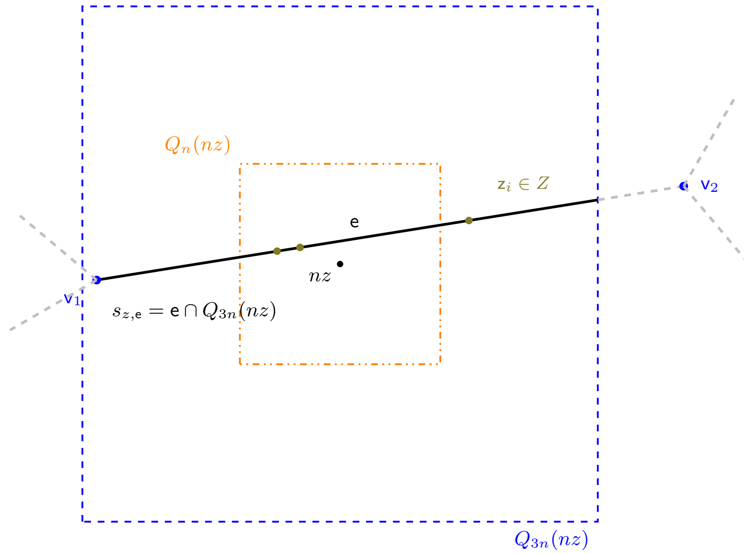

Figure 5: The coarse-graining construction for the proof of Theorem 4. The street edge e (bold) crosses the inner box $Q_n(nz)$, and the truncated segment $s_{z,\mathsf{e}} = \mathsf{e} \cap Q_{3n}(nz)$ is the portion within the outer box $Q_{3n}(nz)$. The points $\mathsf{z}_i \in Z$ are users on the street. The vertices v_1 and v_2 are degree-3 crossroads with relays.

5.2 Proof of Theorem 4

Let the interference factor $\eta > 0$ be fixed. We want to show that the signal-to-interference graph $\mathcal{G}$ does not percolate when the relay probability $p = 1$ and the user intensity λ is sufficiently high. In other words, even when there are relays at all the crossroads (with probability one), we want to show percolation does not occur when there are too many users on the streets. We again use a coarse-graining argument by introducing a percolation model on the integer lattice $\mathbb{Z}^2$ constructed in such a way that if it does not percolate, then neither does the graph $\mathcal{G}$. We then prove that the lattice model does not percolate by exploiting its local dependence.

Definition 5.4. For $n \geq 1$, say a site $z \in \mathbb{Z}^2$ is *n-good* if the following conditions are satisfied:

A. Stabilization radius $R(Q_{3n}(nz)) < 3n$;

B. For all edges $\mathsf{e} \in \mathcal{E}$ if $\mathsf{e} \cap Q_n(nz) \neq \emptyset$, then the segment $s_{z,\mathsf{e}} := \mathsf{e} \cap Q_{3n}(nz)$ is closed.

Say a site $z \in \mathbb{Z}^2$ is *n-bad* if it is not n-good.

Note that if an edge e is an open edge, any segment s in e that contains at least two points of the point process Z is open. Also note that, assuming $\mathsf{e} \cap Q_n(nz) \neq \emptyset$, if $n > r$ (where $r = \ell^{-1}(\tau N)$), then if e is open, there are at least two points of Z in $\mathsf{e} \cap Q_{3n}$.

Lemma 4.6. *Percolation of $\mathcal{G}$ implies, almost surely, percolation of the process of n-bad sites with $n > r = \ell^{-1}(\tau N)$.*

Proof. Assuming that the graph $\mathcal{G}$ percolates, we denote by $\mathcal{C}$ an unbounded (connected) component of $\mathcal{G}$. Furthermore, we define the component $\mathcal{Z} = \mathcal{Z}_n := \{z \in \mathbb{Z}^2 : \mathcal{C} \cap Q_n(nz) \neq \emptyset\}$. Since the component $\mathcal{C}$ is unbounded we have $\#(\mathcal{Z}) = \infty$.

Referring to Definition 5.4, for any point $z \in \mathcal{Z}$, we argue the point z is n-bad because Condition B. of n-goodness is not satisfied owing to the fact there exists an open street intersecting $Q_n(nz)$; refer to Figure 5. We do this by arguing that the segment $s_{z,\mathsf{e}}$ is open because it is long enough to have two points $\mathsf{z}_i, \mathsf{z}_j \in Z$. For the street e, consider one of its street ends $\mathsf{v}_{\mathsf{e},a}$ and, for a reference, a non-vertex point $M \in Q_n \cap s_{z,\mathsf{e}}$. Define the subsegment $s_{a,M} = [M, \mathsf{v}_{\mathsf{e},a}]$. Either the street end $\mathsf{v}_{\mathsf{e},a}$ is inside Q_{3n}, which means there is one point z_i on the segment $s_{a,M}$, or the street end is outside of Q_{3n}, which means the segment $s_{a,M} \cap (Q_{3n} \setminus Q_n)$ is longer than n. Since the street is open, there must be at least one point z_i on this part of the segment due to requirement $n > r = \ell^{-1}(\tau N)$. The same logic applies to other street end $\mathsf{v}_{\mathsf{e},b}$, so there is a point z_j on the subsegment $s_{M,b} = [\mathsf{v}_{\mathsf{e},b}, M]$.

Now observe that the component $\mathcal{Z}$ is almost surely connected in $\mathbb{Z}^2$ in the sense that, for two points $z, z' \in \mathbb{Z}^2$, where $z \neq z'$, are connected in the lattice $\mathbb{Z}^2$ if $\|z - z'\|_1 = 1$). This follows from the fact that the probability that some edge $\mathsf{e} \in \mathcal{E}$ of the Poisson-Voronoi tessellation intersects the lattice $\mathbb{Z}^2$ is equal to zero. (This is true for the Voronoi tessellation generated by any stationary point process; see, for example, the lectures [2, Lemma 11.2.3]). Hence, the point process of n-bad sites percolates. □

Lemma 4.6 implies that it enough to prove that the point process of n-bad sites does not percolate (for some n) when the user intensity λ is sufficiently high.

For our proof we will use the fact that our site model is a 13-dependent percolation model on the integer lattice $\mathbb{Z}^2$.

Lemma 4.7. *For $z \in \mathbb{Z}^2$, set $\zeta_z \coloneqq \mathbb{1}\{z \text{ is } n\text{-bad}\}$. Then $(\zeta_z)_{z\in\mathbb{Z}^2}$ is a 13-dependent random field.*

Proof. We first note that for all $z \in \mathbb{Z}^2$, we have $\zeta_z = 1 - \mathbb{1}\{z \text{ is } n\text{-good}\}$. It is therefore equivalent to prove that the process of n-good sites is 13-dependent.

For $z \in \mathbb{Z}^2$, we set the random variables $\xi_z = \mathbb{1}\{z \text{ is } n\text{-good}\}$. Let the collection of points $\{z_1, \dots z_q\} \subset \mathbb{Z}^2$ be such that for all $i \neq j$, we have $\|z_i - z_j\|_\infty > 13$. We want to show that the random variables $(\xi_{z_i})_{1\le i\le q}$ are independent. This is equivalent to showing that

$$\mathbf{E}\left(\prod_{i=1}^{q} \xi_{z_i}\right) = \prod_{i=1}^{q} \mathbf{E}(\xi_{z_i}), \tag{76}$$

due to the fact that we are dealing with indicator functions.

The above product expression now becomes

$$\mathbf{E}\left(\prod_{i=1}^{q} \xi_{z_i}\right) \tag{77}$$

$$= \mathbf{E}\left[\mathbf{E}\left(\prod_{i=1}^{q} \xi_{z_i}\middle|\Lambda\right)\right] \tag{78}$$

$$= \mathbf{E}\left[\mathbf{E}\left(\prod_{i=1}^{q} \mathbb{1}\{R(Q_{3n}(nz_i)) < 3n\} \prod_{i=1}^{q} \mathbb{1}\{A_{z_i}\}|\Lambda\right)\right] \tag{79}$$

$$= \mathbf{E}\left[\prod_{i=1}^{q} \mathbb{1}\{R(Q_{3n}(nz_i)) < 3n\}\mathbf{E}\left(\prod_{i=1}^{q} \mathbb{1}\{A_{z_i}\}|\Lambda\right)\right], \tag{80}$$

where

$$A_{z_i} = \{\forall e \in \mathcal{E} : e \cap Q_n(nz_i) \neq \emptyset, s_{z_i,e} \text{ is closed}\} : \quad 1 \le i \le q$$

with $s_{z_i,e} = e \cap Q_{3n}(nz_i)$, and where we have used the Λ-measurability of the random variables $\{R(x)\}_{x\in\mathbb{R}^2}$ in line (80).

According to Definition 4.4, the event A_{z_i} only depends on the configuration of the random measure Λ and of the Cox point process X^λ inside the cube $Q_{3n}(nz_i)$. Hence, given the measure Λ, the events $\{A_{z_i} : 1 \le i \le q\}$ only depend on $X^\lambda \cap Q_{3n}(nz_i)$ for $1 \le i \le q$. Then the cubes $Q_{3n}(nz_i)$ are disjoint, owing to the fact $\|z_i - z_j\|_\infty > 13$ for all $i \neq j$.

Moreover, given the measure Λ, the user point process X^λ is distributed according to a Poisson point process. Thus, by Poisson independence property, the events $(A_{z_i})_{1\le i\le q}$ are conditionally independent given the measure Λ. Hence, equation (80) gives

$$\mathbf{E}\left(\prod_{i=1}^{q} \xi_{z_i}\right) = \mathbf{E}\left[\prod_{i=1}^{q} \mathbb{1}\{R(Q_{3n}(nz_i)) < 3n\} \prod_{i=1}^{q} \mathbf{E}\left(\mathbb{1}\{A_{z_i}\}|\Lambda\right)\right]. \tag{81}$$

Recall that $\Lambda_{Q_n(x)}(\cdot) =: \Lambda(Q_n(x) \cap \cdot)$ denotes the restriction of the random measure Λ to the cube $Q_n(x)$. We set the function

$$f(\Lambda_{Q_{3n}(x)}) \coloneqq \mathbf{E}\left(\mathbb{1}\{A_{z_i}\} | \Lambda \right). \tag{82}$$

Then the function f is a deterministic, bounded and measurable function of $\Lambda_{Q_{3n}(x)}$. Moreover, the set $\varphi \coloneqq \{nz_1, \ldots, nz_q\} \subset \mathbb{R}^2$ is a finite subset of $\mathbb{R}^2$ such that for all $i \neq j$ it satisfies the condition

$$\|nz_i - nz_j\|_\infty > 13n > 9\sqrt{2}n\,. \tag{83}$$

Hence for all $i \neq j$, using the Euclidean norm, we have the condition $\|nz_i - nz_j\| > 9n$, and so for all $x \in \varphi$, the set φ satisfies

$$\mathrm{dist}_2(x, \varphi \setminus \{x\}) > 9n\,. \tag{84}$$

Hence, recalling our stabilization result stated in Proposition 4.2, Property 3 implies that the random variables appearing on the right-hand side of equation (81) are all independent. This yields the final expression

$$\begin{aligned} \mathbf{E}\left(\prod_{i=1}^{q} \xi_{z_i} \right) &= \prod_{i=1}^{q} \mathbf{E}\left(\mathbb{1}\{R(Q_{3n}(nz_i)) < 3n\} \prod_{i=1}^{q} \mathbf{E}\left(\mathbb{1}\{A_{z_i}\} | \Lambda \right) \right) \\ &= \prod_{i=1}^{q} \mathbf{E}\left(\xi_{z_i}\right)\,, \end{aligned}$$

which completes the proof. □

For finite n, we now prove that we can make arbitrarily small the probability of a site at the origin $\mathcal{O}$ being n-bad by first taking a large enough n and then a high enough user intensity λ. We formally state this in the next result.

Lemma 4.8. *There exists the limit*

$$\lim_{n\uparrow\infty} \lim_{\lambda\to\infty} \mathbb{P}(\mathcal{O} \text{ is } n\text{-bad}) = 0\,. \tag{85}$$

Proof. We write the truncated segment $s_{0,\mathsf{e}} \coloneqq \mathsf{e} \cap Q_{3n}(\mathcal{O})$. Then we arrive at the probability bound

$$\begin{aligned} &\mathbb{P}(\mathcal{O} \text{ is } n\text{-bad}) \\ &\quad = \mathbb{P}\Big(\{R(Q_{3n}(\mathcal{O})) \geq 3n\} \cup \{\exists\, e \in \mathcal{E} :\, e \cap Q_{3n}(\mathcal{O}) \neq \emptyset \text{ and } s_{0,e} \text{ is open}\} \Big) && (86) \\ &\quad \leq \mathbb{P}\left(R(Q_{3n}(\mathcal{O}) \geq 3n\right) + \mathbb{P}\left(\exists\, e \in \mathcal{E} :\, e \cap Q_{3n}(\mathcal{O}) \neq \emptyset \text{ and } s_{0,e} \text{ is open}\right) && (87) \\ &\quad \leq \mathbb{P}\left(R(Q_{3n}(\mathcal{O})) \geq 3n\right) && (88) \\ &\qquad + \mathbb{P}\left(\exists\, e \in \mathcal{E} :\, e \cap Q_n(\mathcal{O}) \neq \emptyset \text{ and } s_{0,e} \text{ is open}\right). && (\text{b}) \end{aligned}$$

We can now take any $\epsilon > 0$. The stabilization property of the Poisson-Voronoi tessellation (Proposition 4.2) gives the limit $\lim_{n\uparrow\infty} \mathbb{P}(R(Q_n(\mathcal{O})) \geq n) = 0$, then clearly $\lim_{n\uparrow\infty} \mathbb{P}(R(Q_{3n}(\mathcal{O})) \geq 3n) = 0$. This means that we can fix n large enough to make the probability $\mathbb{P}\left(R(Q_{3n}(\mathcal{O})) \geq 3n\right)$ in the above probability bound smaller than $\epsilon/2$.

Our next result tells us that for given n, we can make the user intensity λ large enough to make the probability $\mathbb{P}\left(\exists\, e \in \mathcal{E} : s_{0,e} \text{ is open}\right)$ in the above probability bound smaller than $\epsilon/2$.

Lemma 4.9. *For all $\epsilon > 0$, there exists a critical user intensity λ_c such that if $\lambda > \lambda_c$, the probability in (b) smaller than $\epsilon/2$.*

Proof. If there is no edge intersecting Q_n, then the probability in (b) is 0. Otherwise, let e be such an edge. Let $(\mathsf{z}_i)_{i\in\{1,\ldots,m\}}$ be the points of Z on the street segment $s_{0,e}$. If $m < 2$ then $s_{0,e}$ is closed. Otherwise, take z_1 at either extremity and order z_i in such a way that for all i, we have $\|\mathsf{z}_1 - \mathsf{z}_i\| < \|\mathsf{z}_1 - \mathsf{z}_{i+1}\|$. Let $i_0 < m$ and

$$d := \|\mathsf{z}_{i_0+1} - \mathsf{z}_{i_0}\| \,, \tag{89}$$

then by equation (1), we have the fact

$$\mathrm{STINR}(\mathsf{z}_{i_0+1}, \mathsf{z}_{i_0}) = \frac{(1/N)\ell(d)}{1 + (\eta/N) \sum_{k \neq i_0} \ell(||\mathsf{z}_k - \mathsf{z}_{i_0}||)} \,, \tag{90}$$

and since $s_{0,e} \subset Q_{3n}$, then $||\mathsf{z}_k - \mathsf{z}_{i_0}|| < 3\sqrt{2}\, n$ so that:

$$\sum_{k \neq i_0} \ell(||\mathsf{z}_k - \mathsf{z}_{i_0}||) \geq (m-1)\ell(3\sqrt{2}n) \,, \tag{91}$$

which then carries on through to the signal-to-interference values

$$\mathrm{STINR}(\mathsf{z}_{i_0+1}, \mathsf{z}_{i_0}) \leq \frac{(1/N)\ell(0)}{1 + (\eta/N)(m-1)\ell(3\sqrt{2}n)} \tag{92}$$

where we majored $\ell(d)$ by $\ell(0)$ since ℓ is a bounded decreasing function.

For the signal-to-total-interference-plus-noise ratio to be below threshold τ, we require the inequality:

$$\frac{(1/N)\ell(0)}{1 + (\eta/N)(m-1)\ell(3\sqrt{2}n)} < \tau \,. \tag{93}$$

Rearranging gives:

$$m > M := 1 + \frac{1}{\eta\ell(3\sqrt{2}n)} \left[\frac{\ell(0)}{N\tau} - 1 \right] \,. \tag{94}$$

Since the segment $s_{0,e} \subset Q_{3n}$ has length at most $\mathrm{diam}(Q_{3n}) = 6\sqrt{2}n$, the number of users m on the segment is Poisson distributed with mean at most $\lambda \cdot 6\sqrt{2}n$. By Poisson tail bounds, for $\lambda > M/(6\sqrt{2}n)$, we have $\mathbb{P}(m > M) \to 1$ as $\lambda \to \infty$.

Therefore, for given n, choosing λ large enough ensures $\mathbb{P}(m \leq M) < \epsilon/4$, which in turn guarantees $\mathbb{P}[\mathrm{STINR}(\mathsf{z}_{i_0+1}, \mathsf{z}_{i_0}) < \tau] \geq 1 - \epsilon/4$.

For given n, large enough user intensity λ makes m stochastically large, and we have the probability bound

$$\mathbb{P}[\mathrm{STINR}(\mathsf{z}_{i_0+1}, \mathsf{z}_{i_0}) \geq \tau] < \epsilon/2 \,, \tag{95}$$

which completes the proof of Lemma 4.9. □

The above proof of Lemma 4.9 also completes the proof of Lemma 4.8. □

By Lemma 4.7 and Lemma 4.8, and Proposition 4.1 (where we now consider occupied sites as the n-bad sites), given $\eta > 0$, for large enough λ and for large enough $n < \infty$, the process of n-bad sites is stochastically dominated from above by an independent site percolation model on the integer lattice where the probability of having an open site is arbitrarily small. Hence, this independent site percolation model is subcritical. Consequently, we can make the process of n-bad sites non-percolating. By Lemma 4.6 the same is also true for the signal-to-interference graph $\mathcal{G}$, thus completing the proof of Theorem 4.

A Relationship between SINR and STINR

We consider a finite collection of transmitter-and-receivers $\{z_1, \ldots, z_n\}$ located in $\mathbb{R}^2$. We define the *signal-to-interference-plus-noise ratio (SINR)* at z_j with respect to the incoming signal from a transmitter at z_i as

$$\mathrm{SINR}(z_i, z_j) := \frac{P_{i,j}}{N + \eta I_{i,j}} \,, \quad z_i \neq z_j \,, \tag{96}$$

where we recall that $N \geq 0$ is a noise constant, $P_{i,j}$ is the power of a signal received at z_j originating from a transmitter at z_i, and η is the interference factor, and we introduce

$$I_{i,j} = \sum_{k \neq i,j} P_{k,j} \tag{97}$$

which is the *interference*, meaning the sum of signals from transmitters $\{z_1, \dots, z_n\} \setminus \{z_i, z_j\}$. In general, the signal-to-interference is not a symmetric function, meaning $\mathrm{SINR}(z_i, z_j) \neq \mathrm{SINR}(z_j, z_i)$.

We rewrite the signal-to-interference-plus-noise ratio, given by equation (96), as

$$\mathrm{SINR}(z_i, z_j) := \frac{P_{i,j}}{N + \eta[I_j - P_{i,j}]}\,, \quad z_i \neq z_j\,. \tag{98}$$

For $z_i \neq z_j$, observe

$$\frac{\mathrm{STINR}(z_i, z_j)}{1 - \eta\,\mathrm{STINR}(z_i, z_j)} = \frac{P_{i,j}/(N + \eta I_j)}{1 - \eta P_{i,j}/(N + I_j)} \tag{99}$$

$$= \frac{P_{i,j}}{(N + \eta I_j) - \eta P_{i,j}} \tag{100}$$

$$= \mathrm{SINR}(z_i, z_j)\,. \tag{101}$$

Hence, we have the relations

$$\mathrm{SINR}(z_i, z_j) = \frac{\mathrm{STINR}(z_i, z_j)}{1 - \eta\,\mathrm{STINR}(z_i, z_j)}\,, \quad z_i \neq z_j\,, \tag{102}$$

and

$$\mathrm{STINR}(z_i, z_j) = \frac{\mathrm{SINR}(z_i, z_j)}{1 + \eta\,\mathrm{SINR}(z_i, z_j)}\,, \quad z_i \neq z_j\,. \tag{103}$$

For threshold values $\tau, \tau' \geq 0$, observe the equivalence relationship

$$\mathrm{SINR}(z_i, z_j) \geq \tau' \iff \mathrm{STINR}(z_i, z_j) \geq \tau\,, \quad z_i \neq z_j\,, \tag{104}$$

where

$$\tau = \frac{\tau'}{1 + \eta\tau'} \tag{105}$$

$$\tau' = \frac{\tau}{1 - \eta\tau}\,. \tag{106}$$

Using the signal-to-total-interference is often more natural [3, 19].

B Proof of stabilization

f Property 1 in Proposition 4.2 holds for the new stabilization radius R given in (4.1). As for Property 3 in Proposition 4.2, following Remark 4.1, for any $y \in Q_n(x) \cap \mathbb{Q}^2$, stabilization radius $R^*(y) \leq R(y)$. Since stabilization Property 3 holds with the stabilization radius $R^*(y)$, it is also true for the radius $R(y)$. Consequently, we will now focus on Property 2 in Proposition 4.2. Due to Definition 4.1, we only have to consider the radius $R(y)$ for points $y \in \mathcal{V}$.

The bulk of the proof of Proposition 4.2 hinges upon the next claim.

Claim 4.1. *Let $\{\square_i\}_i$ be a partition of boxes of width $\Delta = n/(3\sqrt{2})$ covering Q_{2n}. If for some point $y \in Q_n \cap \mathcal{V}$ its stabilization radius $R(y) > n$, then there exists a square $\{\square_{i_0}\} \in \{\square_i\}_i$ that contains no points of the point process Φ.*

For clarity, we will prove this result later. Then, provided the above claim is true, we have the bound

$$\begin{aligned}
&\mathbb{P}\left[\max_{y\in Q_n\cap\mathcal{V}} R(y) > n\right] \\
&= \mathbb{P}\left[\text{There exists } y \in Q_n\cap\mathcal{V} \text{ such that } R(y) > n\right] && (107)\\
&\le \mathbb{P}\left[\text{There exists a square } \{\square_{i_0}\} \in \{\square_i\}_i \text{ such that } \Phi(\square_{i_0}) = 0\right] && (108)\\
&\le 81 e^{-\mu\Delta^2}\,. && (109)
\end{aligned}$$

Then clearly we have the limit

$$\lim_{n\to\infty} \mathbb{P}\left[\max_{y\in Q_n} R(y) > n\right] = 0\,, \tag{110}$$

which demonstrates Property 2 in Proposition 4.2.

We now prove Claim 4.1.

Proof. Claim 4.1 Referring to Figure 3, for a given vertex v_1, we refer to the vertices neighboring v_1 as vertices $\mathsf{v}_2, \mathsf{v}_3, \mathsf{v}_4$, whereas we refer to the vertices neighboring $\mathsf{v}_2, \mathsf{v}_3$ and v_4 as vertices $\mathsf{v}_5, \dots, \mathsf{v}_m$, where the positive integer $m \le 10$ depends on the exact configuration. Let $T_1, \dots, T_m$ be the maximum radii of open balls centered at vertices $\mathsf{v}_1, \dots, \mathsf{v}_m$ such that $\Phi(B_{\mathsf{v}_i}(T_i)) = 0$, meaning for $i = 1, \dots, m$ there are no points of Φ in the ball

$$B_i := B_{\mathsf{v}_i}(T_i)\,. \tag{111}$$

For $i = 2, \dots, m$, define the distance

$$U_i := T_i + \|\mathsf{v}_i - \mathsf{v}_1\|\,. \tag{112}$$

as the distance from vertex v_1 to the farthest point in the ball B_i.

By Definition 4.1, the radius

$$R(v_1) = \sup_{\mathsf{e}: v_1 \in \mathsf{e}} \left[R_2(\mathsf{e}, v_1)\right]\,, \tag{113}$$

which by equation (17) gives the bound

$$R(v_1) \le \max_{2\le i\le m} (U_i) \tag{114}$$

The definition of the distance U_i implies that $U_i \le T_1 + 2T_i$, so the balls B_1 and B_i must intersect, because the vertices v_1 and v_i where the three Voronoi cells meet; see Figure 3. Consequently, if $T_1 \le n/3$ and $T_i \le n/3$, then we have the distance bound $U_i \le n$. Taking the complement, we arrive at

$$U_i > n \Rightarrow T_1 > n/3 \text{ or } T_i > n/3. \tag{115}$$

Now consider the event $R(v_1) > n$, which implies $\max_{2\le i\le k}(U_i) > n$. Since there exists a distance $U_i \in \{U_2, \dots, U_m\}$ such that $U_i > n$, then there exists $T_i \in \{T_1, \dots, T_m\}$ such that $T_1 > n/3$ or $T_i > n/3$.

We first consider the ball $B_{\mathsf{v}_1}(5n/3)$ centered at v_1. The event $\max_{1\le i\le m} T_i > n/3$ implies that there exists a ball $B_{\mathsf{v}}(n/3)$, centered at one of the vertices $\{\mathsf{v}_1, \dots, \mathsf{v}_m\}$ containing no points of Φ, which leads to the next claim.

Claim 4.2. *Given the event* $\max_{1\le i\le m} T_i > n/3$*, for a vertex* v_1*, there exists a point* $x \in \mathbb{R}^d$ *such that there is a ball* $B_x(n/3) \subset B_{\mathsf{v}_1}(5n/3)$ *containing no points of the seeding point process* Φ*.*

Proof. To prove our claim, first assume $T_1 \ge n/3$, then we simply set $x = \mathsf{v}_1$ and $B_x(n/3) = B_{\mathsf{v}_1}(n/3)$. On the other hand, if $T_2 \ge n/3$, $\mathsf{v} = \mathsf{v}_2$, either $d(\mathsf{v}_1, \mathsf{v}_2) \le 2n/3$ and then $x = \mathsf{v}_2$, $B_x(n/3) = B_{\mathsf{v}_2}(n/3)$, or $d(\mathsf{v}_1, \mathsf{v}_2) > 2n/3$ and then, because $T_1 < n/3$ and the ball $B_{\mathsf{v}_1}(T_1)$ intersects $B_{\mathsf{v}_2}(T_2)$, we can choose for x the point of the segment $[\mathsf{v}_1, \mathsf{v}_2]$ such that $d(\mathsf{v}_1, x) = 2n/3$. The conclusion remains the same if $T_3 \ge n/3$ or $T_4 \ge n/3$. Now if $T_5 \ge n/3$ while $T_1 < n/3$ and $T_2 < n/3$ (so that $d(\mathsf{v}_1, \mathsf{v}_2) < 2n/3$), if $d(\mathsf{v}_2, \mathsf{v}_5) \le 2n/3$ then $d(\mathsf{v}_1, \mathsf{v}_5) < 4n/3$ and we can choose $x = \mathsf{v}_5$. Now if $d(\mathsf{v}_2, \mathsf{v}_5) > 2n/3$, we can choose for x the point of the segment $[\mathsf{v}_2, \mathsf{v}_5]$ such that $d(\mathsf{v}_2, x) = 2n/3$. Again, the conclusion remains the same if $T_j \ge n/3$ for $j \in 6, \dots, m$, thus proving Claim 4.2. □

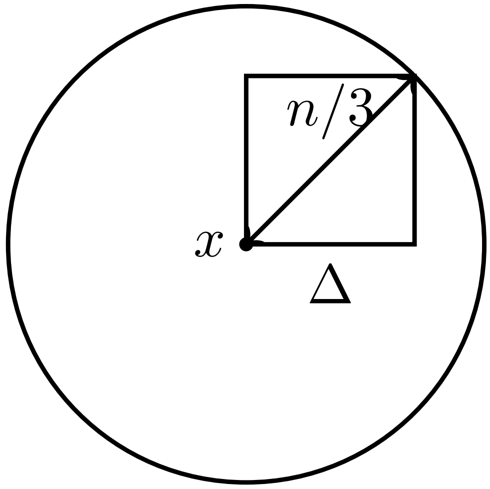

Figure 6: A square $\square$ with width $\Delta = n/(3\sqrt{2})$ containing the center x of the ball $B_x(n/3)$ is necessarily contained within the ball.

To continue the proof of Claim 4.1, which proves Proposition 4.2), we divide Q_{2n} into a partition of boxes $Q_\Delta = \{\square_i\}_i$ of width $\Delta = n/(3\sqrt{2})$ such that $Q_{2n} \subset \cup_{i=1}^{81} \square_i$. Now we can always find a square $\square(\Delta) \subset B_x(n/3)$ with width $\Delta = n/(3\sqrt{2})$ belonging to the partition Q_Δ; see Figure 6.

Consequently, the event

$$R(v_1) > n \tag{116}$$

implies there exists an integer $i \in [1, 81]$ such that $\Phi(\square_i(\Delta)) = 0$. Choosing $v_1 = y$ completes the proof of Claim 4.1. $\square$

The above proof of Claim 4.1 completes the proof of Proposition 4.2. $\square$

C Notation

The following table lists the main notation used throughout the paper. Notation specific to proofs is generally omitted.

Symbol	**Description**
Street Layout (Poisson–Voronoi Tessellation)	
Φ	Homogeneous Poisson point process seeding the Voronoi tessellation
μ	Intensity of the Poisson point process Φ
S	Street layout (Poisson–Voronoi tessellation)
$\mathcal{E} = (\mathsf{e}_i)_{i\geq 1}$	Set of edges (streets) in the street layout S
$\mathcal{V} = (\mathsf{v}_i)_{i\geq 1}$	Set of vertices (crossroads or intersections) in S
e	An edge (street), always considered as a closed connected segment
v	A vertex (crossroads or intersection)
$\mathsf{v}_{\mathsf{e},a}, \mathsf{v}_{\mathsf{e},b}$	The two endpoints (vertices) of edge e
Random Measure and Intensities	
Λ	Random measure on $\mathbb{R}^2$ with $\Lambda(dx) = \nu_1(S \cap dx)$
ν_1	One-dimensional Hausdorff measure on $\mathbb{R}^2$
γ	Street intensity, $\gamma = \mathbf{E}[\Lambda([0,1]^2)] = 2\sqrt{\mu}$

Symbol	Description
Point Processes (Users and Relays)	
X^λ	Cox point process of users with intensity measure $\lambda\Lambda$
λ	User intensity (linear density of users on streets)
Y	Bernoulli point process of relays on vertices $\mathcal{V}$
p	Relay probability (probability a relay occupies a vertex)
$Z = X^\lambda \cup Y$	Superposition of users and relays
$\mathsf{z}_i, \mathsf{z}_j$	Generic points in Z (users or relays)
$Z\|_{\mathsf{e}}$	Restriction of point process Z to edge e, i.e., $Z \cap \mathsf{e}$
Connectivity Model	
$\mathrm{STINR}(\mathsf{z}_i, \mathsf{z}_j)$	Signal-to-total-interference-plus-noise ratio at z_j from z_i
$\mathrm{SINR}(\mathsf{z}_i, \mathsf{z}_j)$	Signal-to-interference-plus-noise ratio at z_j from z_i
$P_{i,j}$	Power of signal received at z_j from transmitter at z_i
I_j	Total interference at point z_j
N	Noise constant (typically $N > 0$)
η	Interference factor ($\eta \geq 0$), technology-dependent parameter
τ	Signal-to-interference threshold, $\tau \in (0, 1/\eta]$ when $\eta > 0$
$\ell(\cdot)$	Path loss function, bounded, continuous, decreasing on $[0, \infty)$
F_i	Fading random variable (set to 1 throughout this work)
$\mathcal{G}$ or $\mathcal{G}_{p,\lambda,\tau,\eta}$	Connectivity graph under signal-to-interference model
$\mathcal{G}^*$ or $\mathcal{G}^*_{p,\lambda,r}$	Connectivity graph under Gilbert model (distance r)
Percolation and Critical Values	
$\theta(p, \lambda, \tau, \eta)$	Percolation probability, $\mathbb{P}[\text{Graph } \mathcal{G} \text{ percolates}]$
$\lambda_{c,1}$	First critical user intensity, $\inf\{\lambda : \theta(p, \lambda, \tau, \eta) > 0\}$
$\lambda_{c,2}$	Second critical user intensity (upper threshold)
Street Neighborhoods and Segments	
$\mathcal{N}(\mathsf{e})$	Street neighborhood of edge e (edge e plus adjacent edges)
s	A segment (closed, connected subset of a street)
$\mathsf{z}_i \sim \mathsf{z}_j$	Points z_i and z_j are neighbors on same street (no points between)
Distances and Conditions	
$d_{\max}(\mathsf{e})$	Maximum distance between neighboring points on edge e
$d_{\min}(\mathcal{N}(\mathsf{e}))$	Minimum distance between neighboring points on neighborhood $\mathcal{N}(\mathsf{e})$
$A_\delta(\mathsf{e})$	Separation condition: $(1/N)\ell(d_{\max}(\mathsf{e})) \geq \tau + \delta$
$B_\delta(\mathcal{N}(\mathsf{e}))$	Clustering condition: $(\eta/N) \sum_{X \in X^\lambda\|_{\mathcal{N}(\mathsf{e})}} \ell(d_{\min}(\mathcal{N}(\mathsf{e}))) \leq \delta/\tau$
Stabilization and Geometric Objects	
$Q_a(x)$	Cube of side a centered at x, i.e., $x + [-a/2, a/2]^2$
Q_a	Cube of side a centered at origin, i.e., $Q_a(\mathcal{O})$
$R(x)$	Stabilization radius at point x, which is newly defined for this work in Definition 4.1
$R^*(x)$	Original stabilization radius, defined as $\inf\{\|x - \mathsf{y}\| : \mathsf{y} \in \Phi\}$ for the Gilbert-style model [23]
$D_1(x)$	First contact distance, $\sup\{r > 0 : \Phi(B_x(r)) = 0\}$
$\psi_1(\mathsf{v})$	Triplet of seed points neighboring vertex v
$\psi_2(\mathsf{e})$	Set of seed points generating the neighborhood of edge e (up to 8 points)
$R_2(\mathsf{e}, x)$	Distance $\sup_{\mathsf{y} \in \psi_2(\mathsf{e})}[\|\mathsf{y} - x\|]$ for $x \in \mathsf{e}$

Symbol	Description
$B_x(r)$	Ball of radius r centered at x
General Notation	
$\mathbb{R}^2$	The Euclidean plane
$\mathbb{Q}^2$	The set of points in $\mathbb{R}^2$ with rational coordinates
$\|\cdot\|$	Euclidean norm in $\mathbb{R}^2$
$\mathbb{P}$ or $\mathbb{P}_{p,\lambda,\tau,\eta}$	Probability measure (with subscripts indicating parameters)
$\mathbf{E}$	Expectation operator
$\perp\!\!\!\perp$	Statistical independence
$\mathbb{1}\{\cdot\}$	Indicator function
support(Λ)	Support of measure Λ

Open/Closed Terminology

- A **crossroads** (vertex) v is **open** if it contains a relay, i.e., $Y(\{\mathsf{v}\}) = 1$.
- A **street** (edge) e is **open** if both endpoints contain relays and these relays are connected (single-hop or multi-hop) along the street.
- A **street neighborhood** $\mathcal{N}(\mathsf{e})$ is **open** if all streets in $\mathcal{N}(\mathsf{e})$ are open.
- A **segment** $s \subset \mathsf{e}$ is **open** if it has at least two points of Z that are multi-hop connected (considering only interference from points in s).

Model Relationships

- **Gilbert model limit**: Setting $\eta \to 0$ (no interference) recovers the Gilbert connectivity model with range $r = \ell^{-1}(\tau N)$.
- **Graph inclusion**: $\mathcal{G}_{p,\lambda,\tau,\eta} \subseteq \mathcal{G}^*_{p,\lambda,r}$ for $r = \ell^{-1}(\tau N)$ (signal-to-interference graph is subgraph of Gilbert graph).
- **Stabilization radii**: $R^*(x) \leq R(x)$ for all $x \in \mathbb{R}^2$, with equality when $x \notin \mathcal{V}$ (non-vertex points).

D Definition index

The following table provides a reference for the main definitions and concepts used throughout the paper.

Concept	Reference
Connectivity model	
Connectivity (single hop)	Definition 2.1
Connectivity graph $\mathcal{G}$	Definition 2.2
Multi-hop connectivity	Definition 2.3
Path loss model	Definition 2.4
Percolation probability θ	Definition 3.1
Open and closed conditions	
Open/closed crossroads	Definition 2.5
Open/closed street	Definition 2.6

Concept	Reference
Open/closed street neighborhood	Definition 4.3
Open/closed street segment	Definition 4.4
Neighboring points on the same street	Definition 4.5
Distance and interference conditions	
Distances $d_{\max}(\mathsf{e})$, $d_{\min}(\mathcal{N}(\mathsf{e}))$	Definition 4.6
Conditions $A_\delta(\mathsf{e})$, $B_\delta(\mathcal{N}(\mathsf{e}))$	Definition 4.7
Stabilization	
Stabilization radius $R(x)$	Definition 4.1
Stabilization radius of a set $R(K)$	Definition 4.2
Stabilization of Λ	Proposition 4.2
Asymptotic essential connectedness of Λ	Proposition 4.3
Coarse-graining and proof machinery	
Bone	Definition 5.1
n-good site (Theorem 3)	Definition 5.2
n-good site (Theorem 4)	Definition 5.4
k-dependence	Definition 5.3
Finite dependent site percolation	Proposition 4.1

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