Some remarks regarding "On the Laplace Transform of the Aggregate Discounted Claims with Markovian arrivals"

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Abstract

Ren [11] considered the problem of deriving the Laplace transform of the aggregate discounted claims in a fixed time period when claims occur according to a Markovian arrival process. Using a martingale argument, he characterized the matrix of Laplace transforms as a solution to a matrix linear differential equation, and then gave the solution. In this note, we point out that the proposed solution of this differential equation is incorrect. It can, however be thought of as a first-order approximation, which appears to be rather accurate.

1 Introduction

Random sums or aggregates are naturally of interest in both actuarial studies and in applied mathematics more generally. Hence there is a strong need for analytic and numerical techniques to evaluate them. Ren [11] presented an appealing model for the aggregate discounted claims in a fixed time period under the assumption that claims occur according to a Markovian arrival process.

He studied a model where claims of size $\{X_k, k \geq 1\}$ occur at a sequence of times $\{T_k, k \geq 1\}$. Assuming that a claim occurring at time t is discounted by a factor $\nu(t)$ such that $0 \leq \nu(t) \leq 1$, the aggregate of the discounted claims at time t is given by the random sum

$$S(t) = \sum_{k=1}^{N(t)} X_k \nu(T_k), \tag{1}$$

with the counting process $\{N(t), \geq t\}$ giving the number of claims in the interval $(0, t]$.

Claims were assumed to occur according to a Markovian arrival process $(N(t), J(t))$ with underlying phase space $\{1, \dots, m\}$ and representation $(\boldsymbol{\gamma}, \mathbf{D}_0, \mathbf{D}_1)$. These point processes, originally introduced as versatile Markovian point processes by Neuts [10], considerably generalize the Poisson point process, explaining their wide use in stochastic models [1, 7]. Ren [11] assumed also that the distribution of the claim size X_k depends on $J(T_k)$, having distribution P_i, density p_i and Laplace transform $\hat{p}(s) = \int_0^\infty p_i(x)e^{-sx}dx$ when $J(T_k) = i$. Consequently, Ren's model for the aggregate $S(t)$ can be thought of as a useful and potentially tractable generalization of classical models, as remarked upon by Li [8] and Shiu [13].

2 The Laplace transform as a solution of a matrix differential equation

For i, $j = 1, 2, \dots, m$, define the Laplace transform

$$L_{i,j}(\xi, t) = \mathbf{E}_i[e^{-\xi S(t)} \mathbb{1}(J(t) = j)], \tag{2}$$

where $\mathbf{E}_i$ denotes the conditional expectation given $\{J(0) = i\}$ and $\mathbb{1}$ is an indicator function. Let $\mathbf{L}(\xi, t)$ be a matrix with i, jth element $L_{i,j}(\xi, t)$. Furthermore, let $\Delta_{\hat{p}}(\xi, t)$ denote a diagonal matrix with $\hat{p}_i(\xi\nu(t))$ as the i th diagonal element.

Ren used an elegant martingale argument to derive the matrix differential equation [11, Equation (2.7)]

$$\frac{\partial \mathbf{L}(\xi, t)}{\partial t} = \mathbf{L}(\xi, t)[\Delta_{\hat{p}}(\xi, t)\mathbf{D}_1 + \mathbf{D}_0] \tag{3}$$

with initial condition $\mathbf{L}(\xi, 0)$ equal to the identity matrix $\mathbf{I}$. He then stated that the solution to (3) is [11, Equation (2.8)]

$$\mathbf{L}(\xi, t) = \exp\left[\int_0^t \Delta_{\hat{p}}(\xi, s)ds\mathbf{D}_1 + \mathbf{D}_0 t\right]. \tag{4}$$

Although there is is a unique solution to the differential equation (3), this solution is not given by (4). It follows that Theorem 2.1 in [11] is not correct.

The intuitive reason for the fact that (4) does not solve (3) is that the integrand in $\int_0^t [\Delta_{\hat{p}}(\xi, s)\mathbf{D}_1 + \mathbf{D}_0]\, ds$, which leads to the the exponential term on the right hand side of (4), does not necessarily commute for different values of s. Put another way, for a square matrix $\mathbf{B}(t)$, in general

$$\frac{d}{dt}[\exp \mathbf{B}(t)] \neq \frac{d\mathbf{B}(t)}{dt}[\exp \mathbf{B}(t)]. \tag{5}$$

The proposed solution (4) holds if $\Delta_{\hat{p}}(\xi, s)\mathbf{D}_1 + \mathbf{D}_0$ commutes for all $s \in (0, t]$, which is clearly the case if $\Delta_{\hat{p}}(\xi, s)$ is independent of s, but is not true in general.

The majority of the further results in [11] were derived via manipulation of the original differential equation (3), and not the proposed solution (4), and we have no reason to doubt these.

3 Expansion techniques

Matrix differential equations of the form

$$\frac{d\mathbf{Y}(t)}{dt} = \mathbf{A}(t)\mathbf{Y}(t), \tag{6}$$

have long been of interest in physics, engineering, and applied mathematics, particularly when a problem can be recast as a linear model. As we mentioned above, this equation has a unique solution, but there is no simple closed-form unless very specific conditions are satisfied. See, for example, Gantmacher [4, Chapter XIV] for a detailed exposition on the theory of such equations.

Numerous expansion or series methods have been proposed for approximating a solution this equation, and there seems to be a divide between research communities according to which methods they prefer. Baake and Schlägel [2] observed that Peano-Baker series are used mostly in engineering books, for example [12, Chapter 3]. Other expansion methods, such as that proposed by Magnus [9], appear mostly in a physics setting; for a thorough survey on Magnus and related expansions; see the survey by Blanes et al. [3], where it is shown that the first term in the the different expansion methods often coincide. Another approach for numerically calculating the solution involves a method known as multiplicative or product integration, which involves taking the limit of infinite products [4, Chapter XIV §6]; also see Gill and Johansen [5].

We focus on the Magnus expansion because the proposed solution (4) turns out to be the first term in this expansion, which in general needs a (possibly infinite) number of terms. To introduce the expansion technique, we use the Lie bracket $[\cdot,\cdot]$, defined by $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$ for two matrices $\mathbf{A}$ and $\mathbf{B}$ of equal size.

Theorem 1 (Magnus 1954)**.** *Let $\boldsymbol{A}(t)$ be a known matrix function of t, and let $\boldsymbol{Y}(t)$ be an unknown matrix function that satisfies the differential equation (6), with initial condition $\boldsymbol{Y}(0) = I$. If certain unspecified convergence conditions are met, then $\boldsymbol{Y}(t)$ can be written in the form*

$$\boldsymbol{Y}(t) = e^{\boldsymbol{\Omega}(T)}, \tag{7}$$

where the matrix function $\boldsymbol{\Omega}(t)$ *has the first terms*

$$\boldsymbol{\Omega}(t) = \int_0^t \boldsymbol{A}_1 dt_1 \tag{8}$$

$$- \frac{1}{2}\int_0^t dt_1 \int_0^{t_1} dt_2 [\boldsymbol{A}_1, \boldsymbol{A}_2] \tag{9}$$

$$+ \frac{1}{6}\int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \left([\boldsymbol{A}_1, [\boldsymbol{A}_2, \boldsymbol{A}_3]] + [[\boldsymbol{A}_1, \boldsymbol{A}_2], \boldsymbol{A}_3]\right) \tag{10}$$

with $\boldsymbol{A}_i := \boldsymbol{A}(t_i)$.

This expansion has been employed, particularly in physics, often with relatively little understanding of conditions for convergence [6]. The calculation of each term, unfortunately, soon becomes unwieldy, although some recursive procedures exist for generating the terms [3, Section 2.3].

4 Counterexamples

In this section we present the results of some numerical experimentation for some cases where there are three phases in the underlying Markovian arrival process and where the claim sizes are exponentially distributed with rates $\mu_1 = 1$, $\mu_2 = 2$ and $\mu_3 = 3$ so that

$$\Delta_{\hat{p}}(\xi, t) = \begin{pmatrix} \frac{1}{1+\xi e^{-\delta t}} & 0 & 0 \\ 0 & \frac{2}{2+\xi e^{-\delta t}} & 0 \\ 0 & 0 & \frac{3}{3+\xi e^{-\delta t}} \end{pmatrix}. \tag{11}$$

We evaluated solutions of the differential equation (3) with standard numerical (Runge-Kutta) techniques in Matlab and compared the results to those given by expression (4). When

$$\mathbf{D}_0 = 2\begin{pmatrix} -4 & 1 & 5 \\ -7 & 1 & 0 \\ 0 & 2 & -5 \end{pmatrix}, \quad \mathbf{D}_1 = 2\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \tag{12}$$

$\xi = 1$, $\delta = 10$ and $t \in [0.13, 0.5]$, the values of the nine entries of the matrix $L_{i,j}(\xi, t)$, calculated according to the numerical solution of (3) and via expression (4), are plotted in Figure 1. We see that the functions do not coincide.

It proved to be surprisingly difficult to construct an example in which the difference between the functions calculated by solving (3)

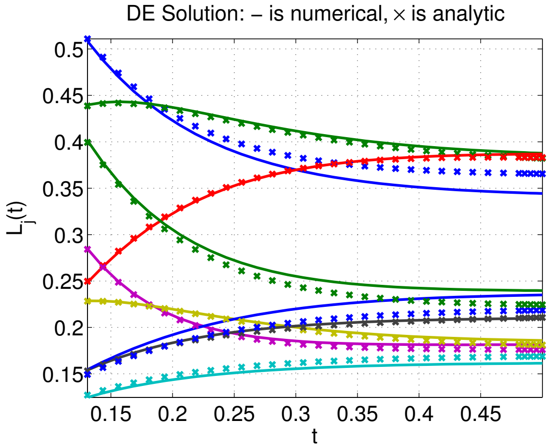

Figure 1: The proposed analytic and numerical solutions for $\xi = 1$ and $\delta = 10$ when $\mathbf{D}_0$ and $\mathbf{D}_0$ are given by (12).

and via expression (4) was discernible. For example, with the same values of ξ and δ,

$$\mathbf{D}_0 = \frac{1}{2}\begin{pmatrix} -4 & 1 & 5 \\ -7 & 1 & 0 \\ 0 & 2 & -5 \end{pmatrix}, \quad \mathbf{D}_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \tag{13}$$

the two functions are almost indistinguishable over the interval $t \in [0, 1]$, see Figure 2. This leads us to speculate that there is a good reason why the first order term (4) in the Magnus expansion is a good approximation of the full solution of the differential equation (3). We make some comments in this direction in the next section.

5 Conclusion

We have pointed out an error in the solution of the matrix differential equation (3). Despite the solution being incorrect, it was difficult for us to produce a counter example where the claimed analytical solution (4) differed substantially from the solution of the differential equation (3). The following argument gives some intuition as to why this should be so.

The integral version of the differential equation (3), which can be derived by an argument analogous to Ren's, is

$$\mathbf{L}(\xi, t) - \mathbf{I} = \int_0^t \mathbf{L}(\xi, u)[\Delta_{\hat{p}}(\xi, u)\mathbf{D}_1 + \mathbf{D}_0]du. \tag{14}$$

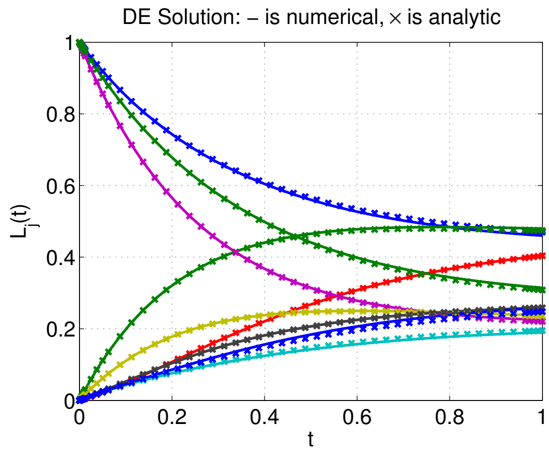

Figure 2: The proposed analytic and numerical solutions for $\xi = 1$ and $\delta = 10$ when $\mathbf{D}_0$ and $\mathbf{D}_0$ are given by (13).

In the practical case where the discount factor is monotonically decreasing with t with $\lim_{t\to\infty} v(t) = 0$, we observe that $\Delta_1 \equiv \Delta_{\hat{p}}(\xi, 0) \leq \Delta_{\hat{p}}(\xi, t) \leq \Delta_{\hat{p}}(\xi, \infty) = I$. Taking the non-negativity of $\mathbf{D}_1$ into account, we see that the right hand side of equation (14) is bounded above by

$$\int_0^t \mathbf{L}(\xi, u)[\mathbf{D}_1 + \mathbf{D}_0]du, \tag{15}$$

and below by

$$\int_0^t \mathbf{L}(\xi, u)[\Delta_1 \mathbf{D}_1 + \mathbf{D}_0]du, \tag{16}$$

and (14) implies that

$$\exp([\Delta_{\hat{p}}(\xi, 0)\mathbf{D}_1 + \mathbf{D}_0]]t) \leq \mathbf{L}(\xi, t) \leq \exp([\mathbf{D}_1 + \mathbf{D}_0]\, t). \tag{17}$$

If the upper and lower bounds (15) and (16) are close as, for example, they are likely to be when ξ is large, we have an explanation of why expression (2.8) of Ren's paper provides a good approximation for the solution of the differential equation (3).

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