A history of two fundamental stochastic processes
Part I: the Poisson (point) process
Part II: the Wiener (or Brownian motion) process

Paul Keeler
Weierstrass Institute for Applied Analysis and Stochastics, Berlin

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John F. C. Kingman laments:

> *In the theory of random processes there are two that are fundamental, and occur over and over again, often in surprising ways. One, the Bachelier-Wiener model of Brownian motion, has been the subject of many books. The other, the Poisson process, seems at first sight humbler and less of worthy of study in its own right....This comparative neglect is ill-judged, and stems from a lack of perception of the real importance of the Poisson process. This distortion is partly comes about from a restriction to one dimension, the while the theory becomes more natural in a more general context.*

Mathematical definition: Poisson (point) process

A Poisson point process N on $\mathbb{R}^d$ is defined with a non-negative Radon measure Λ and two key properties:
1) Number of points in some (Borel) set $B \subset R^d$ is a Poisson random variable with mean (or intensity) measure $\Lambda = \mathbb{E}[N(B)]$.
2) For any any collection of disjoint sets $B_1, \ldots, B_n$, the numbers of points in the bound sets form independent Poisson variables .

- The special case $\Lambda(B) = \lambda|B|$, where $|B|$ is volume of B (or the Lebesgue measure), gives the homogeneous Poisson process which has points uniformly located on the underlying space.
- Often defined and used on the real-line $\mathbb{R}$ (for example, in queueing theory) or on the plane $\mathbb{R}^2$ (for example, in spatial statistics).
- But who dreamt up the Poisson point process?
- What were the original applications or motivations?

Before the process there was the distribution

- Abraham de Moivre (1667 – 1754), in his book “The Doctrine of Chances”, derived the Poisson “distribution” in 1718, by taking the limit of a binomial distribution, but he did not consider it as a probability distribution.
- Simeon Denis Poisson (1781 – 1840) never discovered the Poisson process (a case of Stigler’s law of Eponymy[1]), but in 1837 he independently derived the Poisson distribution, via binomial limit argument, and did consider it as a distribution.
- There are almost no references to the Poisson distribution in the second half of the 19th century.
- Rediscovered independently by Philipp Ludwig von Seidel (1821 – 1896) in 1876, Ernst Abbe (1840 – 1905) in 1879 (counting blood corpuscules), Marian Smoluchowski (1872 – 1917) in 1904 (counting gas molecules) and “Student” (ie William Gosset) (1876 – 1937) in 1909 (counting particles in the plane).

[1] ‘No scientific discovery is named after its original discoverer’.

Enter Bortkiewicz: Meister der Wahrscheinlichkeitsrechnung

- Ladislaus Bortkiewicz (born 1868 in St Petersburg, died 1931 in Berlin) rescued the name "Poisson distribution" from obscurity .
- "Ladislaus von Bortkiewicz: Statistician, Economist and a European Intellectual" – Härdle and Vogt (2015)
- His former student Emil Julius Gumbel wrote:

  > *He was one of the few representatives of mathematical statistics in Germany and as such a lonely figure, highly respected but rarely understood. His writings stimulated numerous scientists in Germany, in the northern European countries and in Italy, but not in England.*

- Bortkiewicz developed his "law of small numbers" and studied deaths by horse-kick in the Prussian army (and child suicide rates) in 1898 in "Das Gesetz der kleinen Zahlen", which cites Poisson[2].

[2] See Haight (1967) for history of the Poisson distribution

Confirmed discoveries of the Poisson process

Real line:

- 1903: Filip Lundberg (1876 – 1965), working on risk and actuarial problems, derived in his thesis an equation, which in the special case becomes the forward equation for the Poisson process, later derived by Andrey Kolmogorov (1903 – 1987) in 1931. (Lundberg's equation also gives the compound Poisson process) [3].
- 1909: A.K. Erlang (1878 – 1929) derived the Poisson distribution (via the binomial limit) while developing a model for the number of incoming phone calls in finite time period.
- 1910: Harry Bateman (1882 – 1946) derived in a paper by Ernest Rutherford and Hans Geiger, who who were counting alpha-particles, the Poisson distribution as a solution of differential equations (Carl Charlier had already derived a solution, but not in such a context).

Plane:

- 1938: Norbert Wiener (1894 – 1964), motivated by making statistical physics concepts more formal, gives a rigorous definition of the Poisson process in the plane (or higher dimensions), which he refers to as "discrete or Poisson chaos". [4]

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[3] Cramér interprets and praises Lundberg's work in 1969.

[4] See Guttorp and Thorarinsdottir(2012) for history of the Poisson process.

Real line:

- 1909: Norman R. Campbell (1880 – 1949), inspired by the work of Rutherford and Geiger, studied shot-noise (ie Schrotrauschen) and derived Campbell's formula, where the underlying point process was the Poisson process [5] (Hardy did the mathematics, but kept his name off the paper.)
- 1907 or 1915: W. H. Grinsted [6] (1885 – 1959) derived the Poisson distribution in a teletraffic setting, and published his results in 1915, but the results appear in a technical report in 1907 at National Telephone Company, which later became British Telecom.

Plane:

- 1922 : Theodor Svedberg (1884 – 1971) proposed a model, where a spatial Poisson point process is the underlying process, in order to study how plants are distributed in plant communities.
- 1937: Kolmogorov (1903 – 1987) used a spatial Poisson point process to model the formation of crystals in metals.

[5] According to Stirzaker 2000.

[6] Sometimes written Grinstead.

Plane:

- 1767: John Michell (1724 – 1793) considered the probability of any particular star being within one degree of another star, given that "they had been scattered by mere chance, as it might happen."
- 1860: Canadian-born American Simon Newcomb (1835 – 1909), motivated by Michell's problem, derived the Poisson distribution via the binomial limit[7].
- 1895: "The great generality of Poisson processes had been anticipated by Abbe." in Last and Penrose (to be published in 2017).
- English translation [8] of Abbe's work:

> *Suppose objects are distributed randomly in some manner in space, or specific events randomly in time, or specific characteristics randomly within a set of discrete things; and it is required to determine the mean frequency of these objects or events or characteristics through counting in a known volume or time interval or subset. Then the result of a single counting will be more or less than the mean value...*

[7] Stigler (2016) considers this as the Poisson process.

[8] Translation from Seneta (1983).

J. Michael Steele pines:

> *Brownian motion is the most important stochastic process. As a practical tool, it has had profound impact on almost every branch of physical science, as well as several branches of social science. As a creation of pure mathematics, it is an entity of uncommon beauty. It reflects a perfection that seems closer to a law of nature than to a human invention.*

Mathematical definition: Wiener or Brownian motion process

A stochastic process $W_t : \{t \geq 0\}$ is a Wiener or Brownian motion process on the real line if it has the following three properties

- Increments $W_t - W_s$ are normally distributed with mean zero and variance $t - s$ for all $0 \leq s \leq t < \infty$.
- For any finite set of times $t_1, \ldots, t_n$ the increments $W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$ form independent random variables.
- W_t are continuous function of t with probability one.

Normal or Gaussian distribution

- Abraham de Moivre (1667 – 1754) also derived the Gaussian distribution as an approximation to the binomial distribution, but did not interpret it as a distribution (he would not have had the concept of a probability density)
- Carl Gauss (1777 – 1855), working on celestial mechanics, developed his theory of errors and introduced the normal distribution.
- Pierre-Simon Laplace (1749 – 1827) made important contributions, for example, calculating the normalization constant.
- Some have called it the Gauss-Laplacian distribution

The Traditional Story (by physicists)

- 1828: Robert Brown (1773–1858) observed pollen seeds (ie inanimate objects) vibrating in fluid, which caused a large debate for some decades.
- 1859 James Clerk Maxwell (1831 – 1879) introduced randomness into physics (Maxwell's distribution is formed from three Gaussian variables), creating statistical physics.
- Statistical physics was then developed mostly by Ludwig Boltzman (1844 – 1906) and Josiah Willard Gibbs (1939 – 1903).
- 1905: Albert Einstein (1879 – 1955) derived a differential (ie heat) equation to describe the probability of finding a particle in a region of space, using ideas from statistical mechanics (ie kinetic theory of gases).
- 1906: Marian Smoluchowski (1872 – 1917), citing Einstein, wrote that he had independently derived the equivalent results earlier by using a different method.
- 1920s: Norbert Wiener (1894 – 1964), inspired by Einsteins work, used a type of measure theory, developed by Percy Daniell (1889 – 1946), and Fourier analysis to prove the existence of the Wiener process as a mathematical object[9].

[9] See Brush (1968) for a history of the Brownian motion process

The Traditional Story (by financial mathematicians)

- 1900 Louis Bachelier (1870 – 1946) introduced a Brownian motion process to model movements of shares or stocks on the Paris Bourse.
- Commonly thought that Bachelier's work was forgotten for decades, rediscovered in the 1950s by Leonard Savage (1917 – 1971), and then became more popular after Bachelier's thesis was translated in 1964.
- But the work was never forgotten in the mathematical community, as Bachelier published a book on his ideas in 1912, which was cited by mathematicians including Joseph Doob (1910 – 2004), William Feller (1906 – 1970) and Kolomogorov (1903 – 1987) .

The (relatively) Untold Story (by Danish statisticians)

- 1880: Thorvald N. Thiele (1838 – 1910) wrote a paper on the method of least squares, where he uses the Brownian motion process to study the errors of a model in time-series analysis.
- Considered as an early discovery of the statistical method known as Kalman filtering (developed in the 1950s and 1960s), but the work was largely overlooked.
- It is thought that the ideas in Thiele's paper were too advanced to have been under stood by the broader mathematical and statistical community at the time [10].

[10] According to Lauritzen (1981)

Historical question: where did the two fundamental processes first meet?

- What was the first result to connect these two stochastic processes?
- Poisson process (on the real line) are Brownian motions are both examples of Lévy processes.
- Jumps of a Lévy process form a spatial Poisson process (for example, on $\mathbb{R}^+ \times \mathbb{R}$), which is often called a Poisson random measure in the Lévy process literature.
- Who was the first to realize that the jumps form a spatial Poisson process?
- Maybe Paul Lévy (1886 – 1971) in 1934, but the paper is in French.

- Poisson distribution: Haight (1967)
- Poisson point process Stirzaker (2000), Guttorp and Thorarinsdottir (2012), Last and Penrose (Lecture notes 2016, to be published in 2017)
- Abbe and de Moivre's contribution to probability: Seneta (1983)
- Brownian process history: Brush (1968), Jarrow and Protter (2004).
- Thiele's contribution: Hald (1981), Lauritzen (1981)
- General history books: Stigler (1986), Hald (1990) and Hald (1998)

Thank you.