Notes on stochastic processes

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Abstract

A stochastic process is a type of mathematical object studied in mathematics, particularly in probability theory, which can be used to represent some type of random evolution or change of a system. There are many types of stochastic processes with applications in various fields outside of mathematics, including the physical sciences, social sciences, finance and economics as well as engineering and technology. This survey aims to give an accessible but detailed account of various stochastic processes by covering their history, various mathematical definitions, and key properties as well detailing various terminology and applications of the process. An emphasis is placed on non-mathematical descriptions of key concepts, with recommendations for further reading.

1 Introduction

In probability and related fields, a stochastic or random process, which is also called a random function, is a mathematical object usually defined as a collection of random variables. Historically, the random variables were indexed by some set of increasing numbers, usually viewed as time, giving the interpretation of a stochastic process representing numerical values of some random system evolving over time, such as the growth of a bacterial population, an electrical current fluctuating due to thermal noise, or the movement of a gas molecule [120, page 7][51, page 46 and 47][66, page 1]. Stochastic processes are widely used as mathematical models of systems and phenomena that appear to vary in a random manner. They have applications in many disciplines including physical sciences such as biology [67, 34], chemistry [156], ecology [16][104], neuroscience [102], and physics [63] as well as technology and engineering fields such as image and signal processing [53], computer science [15], information theory [43, page 71], and telecommunications [97][11][12]. Furthermore, seemingly random changes in financial and other markets have motivated the extensive use of stochastic processes in finance [146][162][118][142].

Applications and the study of phenomena have in turn inspired the proposal of new stochastic processes. Two examples include the Wiener process or Brownian motion process [1], used by Louis Bachelier to study price changes on the Paris Bourse [84], and the Poisson process, proposed by A.K. Erlang to study the number phone calls occurring in a certain period of time [147]. These two stochastic processes are considered the most important and central in the theory of stochastic processes [120, page 8][144, page 32][58, page 99][147], and were discovered repeatedly and independently in different settings and countries, both before and after Bachelier and Erlang [84][147][73].

The terms "stochastic process" and "random process" are used interchangeably, often with no specific index set [5, page 7][148, page 45][39, page 175][128, page 91][87, page 24]. But often these two terms are used when the index set is, for example, the integers or an interval [17, page 1][66, page 1][71, page 1][39, page 175]. If the index set is n-dimensional Euclidean space, then the collection of random variables is usually called a "random field" instead [103, page 1][66, Page 1][5, page 7][143, page 42]. The term"random function" is also used [112, 163][71, page 21][143, page 42], because a stochastic process can also be interpreted as a random element in a function space [87, page 24][103, page 2]. Furthermore, the values of a stochastic process are not always numbers and can be vectors or other mathematical objects [103, page 1][66, page 1][33, page 120][51, Page 47][60, page 294].

Based on their properties, stochastic processes can be divided into various categories, which include random walks [107], martingales [161], Markov processes [126], Lévy processes [7], random fields [4], and Gaussian processes [109], as well as renewal processes, branching processes, and point processes [89][47]. The study of stochastic processes requires mathematical techniques from probability, calculus, linear algebra, set theory, and topology [78][105][64][48] as well as branches of mathematical analysis such as measure theory, Fourier analysis, and functional analysis [23][33][29]. The theory of stochastic processes is considered to be a very important contribution to mathematics [8, Page 1336] and it continues to be an active topic of research for both theoretical reasons and applications [28][151, see Preface][34, see Preface].

[1] The term "Brownian motion" can refer to the physical process, also known as "Brownian movement", and the stochastic process, a mathematical object, but to avoid ambiguity this article uses the terms "Brownian motion process" or "Wiener process" for the latter in a style similar to, for example, Gikham and Skorokhod [66] or Rosenblatt [128].

2 Definitions

2.1 Stochastic process

A stochastic or random process is a collection of random variables that is indexed by or depends on some mathematical set [120, page 7][66, page 1]. More formally, a stochastic process is defined as a collection of random variables defined on a common probability space $(\Omega, \mathcal{F}, , P)$, where Ω is a sample space, $\mathcal{F}$ is a σ-algebra, and P is a probability measure, and the random variables, indexed by some set T, all take values in the same mathematical space S, which must be measurable with some σ-algebra Σ [103, page 1].

In other words, for a given probability space $(P, \mathcal{F}, \Omega)$ and a measurable space (S, Σ), a stochastic process is a collection of S-valued random variables, which can be written as [60, page 293]:

$$\{X(t) : t \in T\}.$$

Historically, in many problems from the natural sciences a point $t \in T$ had the meaning of time, so $X(t)$ is random variable representing a value observed at time t[31, page 528]. A stochastic process can also be written as $\{X(t, \omega) : t \in T\}$, reflecting that is function of the two variables, $t \in T$ and $\omega \in \Omega$ [103, page 2][110, page 11].

There are others ways to consider a stochastic process, but the above definition is considered the traditional one [126, page 121 and 122][9, page 408]. For example, a stochastic process can be interpreted as a S^T-valued random variable, where S^T is the space of all the possible S-valued functions of $t \in T$ that map from the set T into the space S [126, pages 122][87, page 24 and 25].

2.2 Index set

The set T is called the index set [120, page 7][60, page 294] or parameter set[103, page 1][143, page 93] of the stochastic process. Historically, this set was some subset of the real line, such as the natural numbers [2] or an interval, giving the set T the interpretation of time [51, pages 46 and 47]. In addition to these sets, the index set T can be other linearly ordered sets or more general mathematical sets [51, pages 46 and 47] [23, page 482], such as the Cartesian plane R^2 or n-dimensional Euclidean space, where an element $t \in T$ can represent a point in space [144, page 25][89, page 27]. But in general more results and theorems are possible for stochastic processes when the index set is ordered [143, page 104].

[2] In this article the natural numbers start at zero, so they are $0, 1, 2, 3, \ldots$, and so forth.

2.3 State space

The mathematical space S is called the "state space" of the stochastic process. The exact of mathematical spaces varies and it can be many mathematical objects including the integers, the real line, n-dimensional Euclidean space, the complex plane or other mathematical spaces, which reflects the different values that the stochastic process can take [103, page 1][66, page 1][33, page 120][51, Page 47][60, page 294].

2.4 Sample function

A sample function is a single outcome of a stochastic process, so it is formed by taking a single possible value of each random variable of the stochastic process [103, page 2][60, page 296]. More precisely, if $\{X(t,\omega) : t \in T\}$ is a stochastic process, then for any point $\omega \in \Omega$, the mapping

$$X(\cdot,\omega) : T \to S,$$

is called a sample function, a realization, or, particularly when T is interpreted as time, a sample path of the stochastic process $\{X(t,\omega) : t \in T\}$ [126, pages 121 to 124]. This means that for a fixed $\omega \in \Omega$, there exists a sample function that maps the index set T to the state space S [103, page 2]. Other names for a sample function of a stochastic process include trajectory, path function [23, page 493] or path [119, page 10].

3 Notation

A stochastic process can be written, among other ways, as $\{X(t)\}_{t\in T}$ [33, page 120], $\{X_t\}_{t\in T}$ [9, page 408], $\{X_t\}$[103, page 3], $\{X(t)\}$ or simply as X or $X(t)$, although $X(t)$ is regarded as an abuse of notation [95, page 55]. For example, $X(t)$ or X_t are used to refer to the random variable with the index t, and not the entire stochastic process [103, page 3]. If the index set is $T = [0,\infty)$, then one can write the stochastic process as, for example, $(X_t, t \geq 0)$ [39, page 175].

4 Classifications

A stochastic process can be classified in different ways, for example, by its state space, its index set, or the dependence among the random variables. One common way of classification is by the cardinality of the state space and index set [89, page 26] [60, page 294][144, pages 24 and 25].

4.1 Based on index set

Although the index set T can be quite general, traditionally it was assumed to be a subset of the real line, so it can be interpreted as time [51, pages 46 and 47]. In this case, if the index set T of a stochastic process has a finite or countable number of elements, such as a finite set of numbers, the set of integers, or the natural numbers, then the stochastic process is said to be in discrete time [23, page 482][31, page 527]. If the index set T is some interval of the real line, then time is said to be continuous. The two corresponding processes are referred to as discrete-time or continuous-time stochastic processes [89, page 26][33, page 120][129, page 177].

Discrete-time stochastic processes are considered easier to study because continous-time processes require more advanced mathematical techniques and knowledge, particularly due to the index set being uncountable [99, page 63][93, page 153]. If the index set is the integers, or some subset of them, then the stochastic process can also be called a random sequence[31, page 527].

4.2 Based on state space

If the state space S is the integers or natural numbers, such that $S = \ldots, -2, -1, 0, 1, 2, \ldots$ or $S = 0, 1, 2, \ldots$, then the stochastic process is called a discrete or integer-valued process. If the state space is the real line, so $S = (-\infty, \infty)$, then the stochastic process is referred to as a real-valued stochastic process or a stochastic process with continuous state space. If the state space is n-dimensional Euclidean space, so $S = R^n$, then the stochastic process is called a n-dimensional vector process or n-vector process [60, page 294][89, page 26].

5 Examples of stochastic processes

5.1 Bernoulli process

One of the simplest stochastic processes is the Bernoulli process[60, page 293], which is a sequence of independent and identically distributed (iid) random variables, where each random variable takes either the value one with probability, say, p and value zero with probability $1 - p$. This process can be likened to somebody flipping a coin, where the probability of obtaining a head is p and its value is one, while the value of a tail is zero [60, page 301]. In other words, a Bernoulli process is a sequence of iid Bernoulli random variables **[138, page 3]**[68, page 8], where each coin flip is a Bernoulli trial[82, page 11].

5.2 Random walk

Random walks are stochastic processes that are usually defined as sums of iid random variables or random vectors in Euclidean space, so they are processes that change in discrete time [98, page 347][107, page 1][87, page 136][60, page 383][55, page 277]. But some also use the term to refer to processes that change in continuous time [158], particularly the Wiener process used in finance, which has led to some confusion, resulting in criticism [145, page 454]. There are other various types of random walks, defined on different mathematical objects, such as lattices and groups, and in general they are highly studied and have many applications in different disciplines [158][95, page 81].

A classic example of a random walk is known as the "simple random walk", defined on the integers in discrete time, and is based on a Bernoulli process, where each iid Bernoulli variable takes either the value positive one or negative one. In other words, the simple random walk increases by one with probability, say, p, or decreases by one with probability $1-p$, so index set of this random walk is the natural numbers, while its state space is the integers. If the $p = 0.5$, this random walk is called a symmetric random walk [72, page 88][68, page 71].

5.3 Wiener process

The Wiener process is a stochastic process with stationary and independent increments that are normally distributed based on the size of the increments[126, page 1][95, page 56]. The Wiener process is named after Norbert Wiener, who proved its mathematical existence, but the process is also called the Brownian motion process or just Brownian motion due to its historical connection as a model for Brownian movement in liquids, a physical phenomenon originally observed by Robert Brown [38][8][66, page 21].

Playing a central role in the theory of probability, the Wiener process is often considered the most important and studied stochastic process, with connections to other stochastic processes [120, page 8][126, page 1][60, page471][89, pages 21 and 22][88, Preface][125, page IX][146, page 29]. Its index set and state space are the non-negative numbers and real numbers, respectively, that is $T = [0, \infty)$ and $S = [0, \infty)$, so it has both continuous index set and states space [129, page 186]. But the process can be defined more generally so its state space can be n-dimensional Euclidean space [89, pages 21 an 22][144, page 33][95, page 81].

A sample path of a Wiener process is continuous almost everywhere, but it is not differentiable with probability one. It can be considered a continuous version of the simple random walk [117, page 1 and 3][8]. The process arises as the mathematical limit of other stochastic processes such as certain random walks rescaled [88, page 61][142, page 93], which is the subject of Donsker's theorem or invariance principle, also known as the functional central limit theorem [87, pages 225 and 260][88, page 70][117, page 131].

The Wiener process is a member of some important families of stochastic processes, including Markov processes, Lévy processes and Gaussian processes [126, page 1][8]. The process also has many applications and is the main stochastic process used in stochastic calculus [95][88]. It plays a central role in quantitative finance [8][89, page 340], where it was used, for example, in the Black-Scholes-Merton model [95, page 124]. The process is also used in different fields, including the majority of physical sciences as well as some branches of social sciences, as a mathematical model for various random phenomena [88, page 47][160, page 2][page 29][146].

5.4 Poisson process

The Poisson point process, or often just the Poisson process[94], is a stochastic process that has different forms and definitions [152, pages 1 and 2][47, Chapter 2]. It can be defined as a counting process, which is a process that represents the random number of points or events up to some time. The number of points of the process that are located in some interval from zero to some time is a Poisson random variable. This process, which is also called Poisson counting process, has the natural numbers as its state space and the non-negative numbers as its index set, so $S = 0, 1, 2, \dots$ and $T = [0, \infty)$ [152, pages 1 and 2].

If a Poisson process is defined with a single positive constant, then the process is called a homogeneous Poisson process [152, pages 1 and 2][121, page 241]. The homogeneous Poisson process (in continuous time) is a member of important classes of stochastic processes such as Markov processes and Lévy processes [8].

The homogeneous Poisson process can be defined in different ways. It can be defined on the real line, so the index set $T = (-\infty, \infty)$, and this stochastic process is also called the stationary Poisson process [94, page 38][47, page 19]. If the parameter constant of the Poisson process is replaced with some non-negative integrable function of t, the resulting process is called an inhomogeneous or non-homogeneous Poisson process, where the average density of points of the process is no longer constant [94, page 22]. Serving as a fundamental process in queueing theory, the Poisson process is an important process for mathematical models, where it finds applications for models of events randomly occurring in certain time windows[89, pages 118 and 119][96, page 61].

Defined on the real line, the Poisson process can be interpreted as a stochastic process [8][128, page 94], among other random objects [74, page 10 and 18][40, page 41 and 108]. But the Poisson point process can be defined on the n-dimensional Euclidean space or other mathematical spaces [94, page 11], where it is then often interpreted as a random set or a random counting measure [74, page 10 and 18][40, page 41 and 108]. The Poisson point process is one of the most important objects in probability theory, both for applications and theoretical reasons[147][149, page 1]. But it has been remarked that the Poisson process does not receive as much attention as it should, partly due to it often being considered just on the real line, and not on other mathematical spaces [94, Preface][149, page 1].

5.5 Markov processes and chains

Markov processes are stochastic processes, traditionally in discrete or continuous time, that have the Markov property, which means the next value of the Markov process depends on the current value, but it is conditionally independent of the previous values of the stochastic process. In other words, the behavior of the process in the future is stochastically independent of its behavior in the past, given the current state of the process [138, page 2][131, Preface].

The Brownian motion process and the Poisson process (in one dimension) are both examples of Markov processes[130, pages 235 and 358]. These two processes are Markov processes in continuous time, while random walks on the integers and the Gambler's ruin problem are examples of Markov processes in discrete time [60, pages 373 and 374][89, page 49].

A Markov chain is a type of Markov process that has either discrete state space or discrete index set (often representing time), but the precise definition of a Markov chain varies [9, page 7]. For example, it is common to define a Markov chain as a Markov process in either discrete or continuous time with countable state spaces. In other words, this definition implies that Markov chains are Markov processes with discrete state spaces regardless of the nature of time [120, page 188][89, pages 29 and 30][103, Chapter 6][130, pages 174 and 231]. But it is also common to define a Markov chain as having discrete time regardless of the state space [9, page 7].

Markov processes form an important class of stochastic processes and have applications in many areas [89, page 47][105]. For example, they are the basis for a general stochastic simulation method known as Markov Chain Monte Carlo, which is used for simulating random objects with specific probability distributions, and has found extensive application in Bayesian statistics [132, page 225].

The concept of the Markov property was originally for stochastic processes in continuous and discrete time, but the property has been adapted for other index sets, such as n-dimensional Euclidean space, which results in collections of random variables known as Markov random fields [131, Preface][144, page 27][32, page 253].

5.6 Martingales

Martingale are discrete-time or continuous-time stochastic processes with the property that the expectation of the next value of a martingale is equal to the current value given all the previous values of the process. The exact mathematical definition of a martingale requires two other conditions coupled with the mathematical concept of a filtration, which is related to the intuition of increasing available information as time passes. Martingales are usually defined to be real-valued [95, page 65][88, page 11][161, pages 93 and 94], but they can also be complex-valued [51, pages 292 and 293] or even more general [122].

A symmetric random walk and Wiener process are both examples of martingales, respectively, in discrete and continuous time [88, page 11][95, page 65]. For a sequence of independent and identically distributed random variables $X_1, X_2, X_3, \dots$ with zero mean, the stochastic process formed from the successive partial sums $X_1, X_1 + X_2, X_1 + X_2 + X_3, \dots$ is a discrete-time martingale [146, page 12]. In this aspect, discrete-time martingales generalize the idea of partial sums of independent random variables [77, page 2].

Martingales can also be created from stochastic processes by applying some sutiables transformations, which is the case for the homogeneous Poisson process (on the real line) resulting in a martingale called the compensated Poisson process [88, page 11]. Martingales can also be built from other martingales [146, pages 12 and 13]. For example, there are two known martingales based on the martingale the Wiener process, forming in total three continuous-time martingales [95, page 65][146, page 115].

Martingales mathematically formalize the idea of a fair game [130, page 295], and they were originally developed to show that it is not possible to win a fair game[146, page 11]. But now they are used in many areas of probability, which is one of the reasons they are studied [161, page 94][87, page 96][146, page 11]. Many problems in probability have been solved by finding a martingale in the problem and studying it [139, page 371]. Generally speaking, discrete-time martingales are more intuitive and knowledge of them is required to understand continuous-time martingales [126, page 163]. Martingales will converge, given some conditions on their moments, so they are often used to derive convergence results, due largely to martingale convergence theorems [146, page][77, page 2][68, 336].

Martingales have many applications in certain areas of probability theory such as queueing theory and Palm calculus [13] and other fields such as economics [77, page x] and finance [118]. Martingales also have applications in statistics, but it has been remarked that its use and application are not as widespread as it could be in the field of statistics, particularly statistical inference [80, pages 292 and 293].

5.7 Lévy processes

Lévy processes are types of stochastic processes that can be considered as generalizations of random walks in continuous time [8][21, see Preface]. These processes, which are also called Lévy flights in physics, have many applications in fields such as finance, fluid mechanics, physics and biology [8][7, page 69]. The main defining characteristic of these processes is their stationarity property, so they were known as "processes with stationary and independent increments". In other words, for n non-negatives numbers, $0 \leq t_1 \leq \cdots \leq t_n$, the corresponding

$n-1$ increments

$$X_{t_2} - X_{t_1}, \dots, X_{t_{n-1}} - X_{t_n},$$

are all independent of each other, and the distribution of each increment only depends on the difference in time [8].

A Lévy process can be defined such that its state space is some abstract mathematical space, such as a Banach space, but the processes in general are often defined so that they take values in Euclidean space. The index set is the real line, which gives the interpretation of time. Important stochastic processes such as the Wiener process, the homogeneous Poisson point process (in one dimension), and subordinators are all Lévy processes [8][21, Preface].

5.8 Random fields

A random field is a collection of random variables indexed by a n-dimensional Euclidean space or some manifold. In general, a random field can be considered an example of a stochastic or random process, where the index set is not necessarily a subset of the real line [5, page 7]. But there is a convention that an indexed collection of random variables is called a random field when the index has two or more dimensions [103, page 1][66, page 1][100, page 171]. If the specific definition of a stochastic process requires the index set to be a subset of the real line, then the random field is considered as a generalization of stochastic process [7, page 19].

5.9 Point processes

A point process is a collection of points randomly located on some mathematical space such as the real line, n-dimensional Euclidean space, or more abstract spaces. There are different interpretations of a point process, such a random counting measure or a random set [40, page 108][74, page 10]. Some authors regard a point process and stochastic process as two different objects such that a point process is a random object that arises from or is associated with a stochastic process [47, page 194][44, page 3], though it has been remarked that the difference between point processes and stochastic processes is not clear [44, page 3].

Other authors consider a point process as a stochastic process, where the process is indexed by sets of the underlying space[3] on which it is defined, such as the real line or n-dimensional Euclidean space [89, page 31][58, page 232][134, page 99]. Other stochastic processes such renewal and counting processes are studied in the theory of point processes [47][44]. Sometimes the term “point process” is not preferred, as historically the word “process” denoted an evolution of some system in time, so point process is also called a random point field [40, page 109].

[3] In the context of point processes, the term “state space” can mean the space on which the point process is defined such as the real line[94, page 8][116, Preface], which corresponds to the index set in stochastic process terminology.

6 History

6.1 Early probability theory

Probability theory has its origins in games of chance, which have a long history, with some games being played thousands of years ago [49], but very little analysis on them was done in terms of probability [113, Page 1]. The year 1654 is often considered the birth of probability theory when French mathematicians Pierre Fermat and Blaise Pascal had a written correspondence on probability, motivated by a gambling problem [137][153, Chapter 2][150, Pages 24 to 26]. But there was earlier mathematical work done on the probability of gambling games such as "Liber de Ludo Aleae" by Cardano, written in the 16 th century but posthumously published later in 1663 [18].

Jakob Bernoulli [4] later wrote "Ars Conjectandi", which is considered a significant event in the history of probability theory. Bernoulli's book was published, also posthumously, in 1713 and inspired many mathematicians to study probability [113, Page 56][150, Page 37]. But despite some renown mathematicians contributing to probability theory, such as Pierre-Simon Laplace, Abraham de Moivre, Carl Gauss, Siméon Poisson and Pafnuty Chebyshev [41, 25], most of the mathematical community [5] did not consider probability theory to be part of mathematics until the 20th century [19, 52, 46, 41].

6.2 Statistical mechanics

In the physical sciences, scientists developed in the 19th century the discipline of statistical mechanics, where physical systems, such as containers filled with gases, can be regarded or treated mathematically as collections of many moving particles. Although there were attempts to incorporate randomness into statistical physics by some scientists, such as Rudolf Clausius, most of the work had little or no randomness[155, pages 22 and 23][37, pages 150 and 151]. This changed when in 1859 James Clerk Maxwell contributed significantly to the field, more specifically, to the kinetic theory of gases, by presenting work where he assumed the particles in a gas moved in random directions at a random velocities [155, pages 30 and 31][36, page 243]. The kinetic theory of gases and statistical physics continued to be developed in the second half of the 19th century, with work done chiefly by Clausius, Ludwig Boltzmann and Josiah Gibbs, which would later have an influence on Albert Einstein's model for Brownian movement[38, pages 15 and 16].

[4] Also known as James or Jacques Bernoulli [76, Page 221].

[5] It has been remarked that a notable exception was the St Petersburg School in Russia, where mathematicians led by Chebyshev studied probability theory [19].

6.3 Measure theory and probability theory

In 1900 at the International Congress of Mathematicians in Paris David Hilbert presented a list of mathematical problems, where he asked in his sixth problem for mathematical a treatment of physics and probability involving axioms [25]. Around the start of the 20th century, mathematicians developed measure theory, a branch of mathematics for studying integrals of mathematical functions, where two of the founders were French mathematicians, Henri Lebesgue and Emile Borel [52, 25]. Later in 1925 another French mathematician Paul Lévy published the first probability book that used ideas from measure theory [25].

In 1920s significant and fundamental contributions to probability theory were made in the Sovient Union by mathematicians such as Sergei Bernstein, Alexander Khinchin [6], and Andrei Kolmogorov [46]. Kolmogorov published in 1929 his first attempt at presenting a mathematical foundation, based on measure theory, for probability theory [91, page 33]. Two years later Khinchin gave the first mathematical definition of a stochastic process [50].

6.4 Birth of modern probability theory

In 1933 Andrei Kolmogorov published in German his book on the foundations of probability theory titled *Grundbegriffe der Wahrscheinlichkeitsrechnung*[7], where Kolmogorov used measure theory to develop an axiomatic framework for probability theory. The publication of this book is now widely considered to be the birth of modern probability theory, when the theories of probability and stochastic processes became parts of mathematics [25, 46].

After the publication of Kolmogorov's book, further fundamental work on probability theory and stochastic processes was done by Khinchin and Kolmogorov as well as other mathematicians such as Joseph Doob, William Feller, Maurice Fréchet, Paul Lévy, Wolfgang Doeblin, and Harald Cramér [25, 46]. Decades later Cramér referred to the 1930s as the "heroic period of mathematical probability theory" [46]. But World War II greatly interrupted the development of probability theory, causing, for example, the migration of Feller from Sweden to the United States of America [46] and the death of Doeblin, considered now a pioneer in stochastic processes [111].

6.5 Stochastic processes after World War II

After World War II the study of probability theory and stochastic processes gained more attention from mathematicians, with signification contributions made in

[6] The name Khinchin is also written in (or transliterated into) English as Khintchine [50].

[7] Later translated into English and published in 1950 as Foundations of the Theory of Probability [25]

many areas of probability and mathematics as well as the creation of new areas [46, 114]. Starting in the 1940s, Kiyosi Itô published papers developing the field of stochastic calculus, which involves stochastic integrals and stochastic differential equations based on the Wiener or Brownian motion process [1]. Also starting in the 1940s, connections were made between stochastic processes, particularly martingales, and the mathematical field of potential theory, with early ideas by Shizuo Kakutani and then later work by Joseph Doob [114]. Further work, considered pioneering, was done by Gilbert Hunt in the 1950s, connecting Markov processes and potential theory, which had a significant effect on the theory of Lévy processes and led to more interest in studying Markov processes with methods developed by Itô [21, see Preface][139, page 176][84].

In 1953 Doob published his book "Stochastic processes", which had a strong influence on the theory of stochastic processes and stressed the importance of measure theory in probability [27, 114]. Doob also chiefly developed the theory of martingales, with later substantial contributions by Paul-André Meyer. Earlier work had been carried out by Sergei Berstein, Paul Lévy and Jean Ville, the latter adopting the term martingale for the stochastic process [77, page 1 and 2][56]. Methods from the theory of martingales became popular for solving various probability problems. Techniques and theory were developed to study Markov processes and then applied to martingales. Conversely, methods from the theory of martingales were established to treat Markov processes [114].

Other fields of probability were developed and used to study stochastic processes, with one main approach being the theory of large deviations [114]. The theory has many applications in statistical physics, among other fields, and has core ideas going back to at least the 1930s. Later in the 1960s and 1970s fundamental work was done by Alexander Wentzell in the Soviet Union and Monroe D. Donsker and Srinivasa Varadhan in the United States of America [57], which would later result in Varadhan winning the 2007 Abel Prize [123].

The theory of stochastic processes still continues to be a focus of research, with yearly international conferences on the topic of stochastic processes [8, page 1336][54, see Preface][28][151, see Preface][34, see Preface].

6.6 Discoveries of specific stochastic processes

Although Khintchine gave early mathematical definitions of stochastic processes in the 1930s [50, 157], specific stochastic processes had already been discovered in different settings, such as the Brownian motion process and the Poisson point process [84, 73]. Some families of stochastic processes such as point processes or renewal processes have long and complex histories, stretching back centuries [47, chapter 1].

6.6.1 Bernoulli process

The Bernoulli process, which can serve as a mathematical model for flipping a biased coin, is possibly the first stochastic process to have been studied [60, page 301]. The process is a sequence of independent Bernoulli trials [138, page 3], which are named after Jakob Bernoulli who used them to study games of chance, including probability problems proposed and studied earlier by Christiaan Hugens [76]. Bernoulli's work, including the study of Bernoulli trials, was published in Ars Conjectandi in 1713 [76, pages 223 and 226][108, pages 8 to 10].

6.6.2 Random walks

In 1905, Karl Pearson coined the term "random walk" while posing a problem describing a random walk on the plane, which was motivated by an application in biology, but such problems involving random walks had already been studied in other fields. Certain gambling problems that were studied centuries earlier can be considered as problems involving random walks [158][108, pages 8 to 10]. For example, the problem known as the "Gambler's ruin" is based on a simple random walk [89, page 49][60, page 374], and is an example of a random walk with absorbing barriers [82, page 5][135]. Pascal, Fermat and Huyens all gave numerical solutions to this problem without detailing their methods [76, page 63], and then more detailed solutions were presented by Jakob Bernoulli and Abraham de Moivre [76, page 202].

For random walks in n-dimensional integer lattices, George Plya published in 1919 and 1921 work, where he studied the probability of a symmetric random walk returning to a previous position in the lattice. Plya showed that a symmetric random walk, which has an equal probability to advance in any direction in the lattice, will return to a previous position in the lattice an infinite number of times with probability one in one and two dimensions, but with probability zero in three or higher dimensions [60, page 385][81, page 111].

6.6.3 Wiener process

The Wiener process or Brownian motion process has its origins in three separate fields: statistics, finance and physics [84]. In 1880, Thorvald Thiele wrote a paper on the method of least squares, where he uses the process to study the errors of a model in time-series analysis [75, 106]. The work is now considered as an early discovery of the statistical method known as Kalman filtering, but the work was largely overlooked. It is thought that the ideas in Thiele's paper were too advanced to have been understood by the broader mathematical and statistical community at the time [106].

The French mathematician Louis Bachelier used a Wiener process in his 1900 thesis in order to model price movements on the Paris Bourse, a stock exchange [42], without knowing the work of Thiele [84]. It has been speculated that Bachelier drew ideas from the random walk model of Jules Regnault, but Bachelier did not cite him [86], and Bachelier's thesis is now considered pioneering in the field of financial mathematics [86, 42].

It is commonly thought that Bachelier's work gained little attention and was forgotten for decades until it was rediscovered in the 1950s by the Leonard Savage, and then become more popular after Bachelier's thesis was translated into English in 1964. But the work was never forgotten in the mathematical community, as Bachelier published a book in 1912 detailing his ideas [86], which was cited by mathematicians including Doob, Feller [86] and Kolomogorov [84]. The book continued to be cited, but then starting in 1960s the original thesis by Bachelier began to be cited more than his book when economists started citing Bachelier's work [86].

In 1905 Albert Einstein published a paper where he studied the problem of Brownian motion or movement to explain the seemingly random movements of particles in liquids by using ideas from the kinetic theory of gases. Einstein derived a differential equation, known as a diffusion equation, for describing the probability of finding a particle in a certain region of space. Shortly after Einstein's first paper on Brownian movement, Marian Smoluchowski published work where he cited Einstein, but wrote that he had independently derived the equivalent results by using a different method [38].

Einstein's work later inspired Norbert Wiener in the 1920s [38] to use a type of measure theory, developed by Percy Daniell, and Fourier analysis to prove the existence of the Wiener process as a mathematical object [84].

6.6.4 Poisson process

The Poisson process is named after Siméon Poisson, due to its definition involving the Poisson distribution, but Poisson never studied the process [47, pages 8 and 9][147]. There are a number of claims for early uses or discoveries of the Poisson point process [147][73]. At the beginning of the 20th century the Poisson process would arise independently in different situations [147][73]. In Sweden 1903, in Filip Lundberg published a thesis containing work, now considered fundamental and pioneering, where he proposed to model insurance claims with a homogeneous Poisson process [59][45].

Another discovery occurred in Denmark in 1909 when Agner Krarup Erlang derived the Poisson distribution when developing a mathematical model for the number of incoming phone calls in a finite time interval. Erlang was not at the time aware of Poisson's earlier work and assumed that the number phone calls arriving in each interval of time were independent to each other. He then found the limiting case, which is effectively recasting the Poisson distribution as a limit of the binomial distribution [147].

In 1910 Ernest Rutherford and Hans Geiger published experimental results on counting alpha particles. Their experimental work had mathematical contributions from Harry Bateman, who derived Poisson probabilities as a solution to a family of differential equations, resulting in the independent discovery of the Poisson point process [147].

In general, there were many studies and applications of the Poisson point process, but its early history is complicated, which has been explained by the various applications of the process in numerous fields by biologists, ecologists, engineers and various physical scientists [147].

6.6.5 Markov processes

Markov processes and Markov chains are named after Andrei Markov who studied Markov chains in the early 20 th century. Markov was interested in studying an extension of independent random sequences. In his first paper on Markov chains, published in 1906, Markov showed that under certain conditions the average outcomes of the Markov chain would converge to a fixed vector of values, so proving a weak law of large numbers without the independence assumption [69, pages 464 to 466][32, Preface][79], which had been commonly regarded as a requirement for such mathematical laws to hold [79]. Markov later used Markov chains to study the distribution of vowels in "Eugene Onegin", written by Alexander Pushkin, and proved a central limit theorem for such chains [69, pages 464 to 466].

In 1912 Poincaré studied Markov chains on finite groups with an aim to study card shuffling. Other early uses of Markov chains include a diffusion model, introduced by Paul and Tatyana Ehrenfest in 1907, and a branching process, introduced by Francis Galton and Henry William Watson in 1873, preceding the work of Markov [69, pages 464 to 466][32, page ix]. After the work of Galton and Watson, it was later revealed that their branching process had been independently discovered and studied around three decades earlier by Irénée-Jules Bienaymé [136]. Starting in 1928, Maurice Fréchet became interested in Markov chains, eventually resulting in him publishing in 1938 a detailed study on Markov chains [69, pages 464 to 466][35].

Andrei Kolmogorov developed in a 1931 paper a large part of the early theory of continuous-time Markov processes [91, page 33] [46]. Kolmogorov was partly inspired by Louis Bachelier's 1900 work on fluctuations in the stock market as well as Norbert Wiener's work on Einstein's model of Brownian movement [14, page 5][91, page 33]. He introduced and studied a particular set of Markov processes known as diffusion processes, where he derived a set of differential equations describing the processes [91, page 33][143, page 146]. Independent of Kolmgorov's work, Sydney Chapman derived in a 1928 paper an equation, now called the Chapman-Kolmogorov equation, in a less mathematically rigorous way than Kolmogorov, while studying Brownian movement [20]. The differential equations are now called the Kolmogorov equations [6, Preface] or the Kolmogorov-Chapman equations [91, 57]. Other mathematicians who contributed significantly to the foundations of Markov processes include William Feller, starting in 1930s, and then later Eugene Dynkin, starting in the 1950s [46].

6.6.6 Lévy processes

Lévy processes such as the Wiener process and the Poisson process (on the real line) are named after Paul Lévy who started studying them in the 1930s [8], but they have connections to infinitely divisible distributions going back to the 1920s[21, see Preface]. In a 1932 paper Kolmogorov derived a characteristic function for random variables associated with Lévy processes. This result was later derived under more general conditions by Lévy in 1934, and then Khinchin independently gave an alternative form for this characteristic function in 1937 [46][7, page 67]. In addition to Lévy, Khinchin and Kolomogrov, early fundamental contributions to the theory of Lévy processes were made by Bruno de Finetti and Kiyosi Itô [21, see Preface].

7 Etymology

The word "stochastic" in English was originally used as an adjective with the definition "pertaining to conjecturing", and stemming from a Greek word meaning "to aim at a mark, guess", and the Oxford English Dictionary gives the year 1662 as its earliest occurrence [3]. In his work on probability "Ars Conjectandi", originally published in Latin in 1713, Jakob Bernoulli used the phrase "Ars Conjectandi sive Stochastice", which has been translated to "the art of conjecturing or stochastics" [140, page 5]. This phrase was used, with reference to Bernoulli, by economist and statistician Ladislaus Bortkiewicz [141, page 136] who in 1917 wrote in German the word "stochastik" with a sense meaning random. The term "stochastic process" first appeared in English in a 1934 paper by Joseph Doob [3]. For the term and a specific mathematical definition, Doob cited another 1934 paper, where the term "stochastischer Prozeß"' is used in German by Alexander Khinchin [50][92].

Early occurrences of the word "random" in English with its current meaning, relating to chance or luck, date back to the 16th century, while earlier recorded usages started in the 14 th century as a noun meaning "impetuosity, great speed, force, or violence (in riding, running, striking, etc.)". The word itself comes from a Middle French word meaning "speed, haste", and it is probably derived from a French verb meaning to "to run" or "to gallop". The first written appearance of the term "random process" pre-dates "stochastic process", which the Oxford English Dictionary also gives as a synonym, and was used in an article by Francis Edgeworth published in 1888 [2].

8 Terminology

The definition of a stochastic process varies [61, page 580], but a stochastic process is traditionally defined as a collection of random variables indexed by some set [126, page 121 and 122][9, page 408] . The terms "random process" and "stochastic process" are considered synonyms and are used interchangeably, without the index set being precisely specified [5, page 7][148, page 45][39, page 175][128, page 91][87, page 24][70, page 383]. Both "collection" [103, page 1][148, page 45] or "family" are used [120, page 7][83, page 13], while instead of "index set", sometimes the terms "parameter set" [103, page 1] or "parameter space" [5, page 7][119] are used.

The term "random function" is also used to refer to a stochastic or random process [112, 163][66, page 1][33, page 133], though sometimes it is only used when the stochastic process takes real values [83, page 13][103, page 2]. This term is also used when the index sets are mathematical spaces other than the real line [66, page 1][71, page 1], while the terms "stochastic process " and "random process" are usually used when the index set interpreted as time [17, page 1][66, page 1][page 1][71], and other terms are used such as "random field" when the index set is n-dimensional Euclidean space R^n or a manifold[103, page 1][5, page 7][66, page 1].

9 Further definitions

9.1 Law

For a stochastic process X defined on the probability space $(P, \mathcal{F}, \Omega)$, the law of X is defined as the image measure:

$$\mu = P \circ X^{-1},$$

where is P a probability measure, the symbol $\circ$ denotes function composition and X^{-1} is the pre-imagine of the measurable function or, equivalently, the S^T-valued random variable X, where S^T is the space of all the possible S-valued functions of $t \in T$, so the law of a stochastic process is a probability measure [126, pages 122][87, page 24 and 25][62, page 571][124, pages 40 and 41].

For a measurable subset B of S^T, the pre-image of X gives

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\},$$

so the law of a X can be written as [103, page 2]:

$$\mu(B) = P(\{\omega \in \Omega : X(\omega) \in B\}).$$

The law of a stochastic process or a random variable is also called the probability law, probability distribution, or the distribution [159, page 23][7, page 4][125, page 10][62, page 571][31, page 528].

9.2 Finite-dimensional probability distributions

For a stochastic process X with law μ, its finite-dimensional distributions are defined as:

$$\mu_{t_1,\ldots,t_n} = P \circ (X(t_1), \ldots, X(t_n))^{-1},$$

where $n \geq 1$ is a counting number and each set t_i is a non-empty finite subset of the index set T, so each $t_i \subset T$, which means that $t_1, \ldots, t_n$ is any finite collection of subsets of the index set T [87, page 25][126, page 123].

For any measurable subset C of the Cartesian power $S^n = S_1 \times \cdots \times S_n$, the finite-dimensional distributions of a stochastic process X can be written as[103, page 2]:

$$\mu_{t_1,\ldots,t_n}(C) = P(\{\omega \in \Omega : X_{t_1}(\omega), \ldots, X_{t_n}(\omega) \in C\}.$$

The finite-dimensional distributions of a stochastic process satisfy two mathematical conditions known as consistency conditions [129, pages 177 and 178].

9.3 Stationarity

A stochastic process is said to have the property of stationarity when all the random variables of the stochastic process are identically distributed. In other words, if X is a stationary stochastic process, then for any $t \in T$ the random variable X_t has the same distribution, which means that for any set of n index set values $t_1, \ldots, t_n$, the corresponding n random variables

$$X_{t_1}, \ldots X_{t_n},$$

all have the same probability distribution. The index set of a stationary stochastic process is usually interpreted as time, so it can be the integers or the real line [103, page 6 and 7][66, page 4]. But the concept of stationarity also exists for point processes and random fields, where the index set is not interpreted as time[4, page 15][40, page 112] [103, pages 6 and 7].

When the index set T can be interpreted as time, a stochastic process is said to be stationary if its finite-dimensional distributions are invariant under translations of time. This type of stochastic process can be used to describe a physical system that is in steady state, but still experiences random fluctuations [103, pages 6 and 7]. The intuition behind such stationarity is that as time passes the distribution of the stationary stochastic process remains the same [51, pages 94 to 96]. A sequence of random variables forms a stationary process if and only if the random variables are identically distributed [103, pages 6 and 7].

A stochastic process with above definition of stationarity is sometimes said to be strictly stationary, but there are other forms of stationarity. One example is when a discrete-time or continuous-time stochastic process X is said to be stationary in the wide sense, then the process X has a finite second moment for all $t \in T$ and the covariance of the two random variables X_t and X_{t+h} depends only on the number h for all $t \in T$ [51, pages 94 to 96][60, pages 298 and 299]. The concept of stationarity in the wide sense was introduced by Khinchin and has other names including covariance stationarity or stationarity in the broad sense[66, page 8][60, pages 298 and 299].

9.4 Increment

For a discrete-time or continuous-time stochastic process, an increment is how much a stochastic process changes over a certain time period, so it is a random quantity formed by the difference of two random variables of the same stochastic process at two points in time. For example, if X is a continuous-stochastic process with state space S, then for any two non-negative numbers $t_1 \in [0, \infty)$ and $t_2 \in [0, \infty)$ such that $t_1 \leq t_2$, the difference $X_{t_2} - X_{t_1}$ is a S-valued random variable known as an increment [89, page 27] [8]. Often the state space S is the real line or the natural numbers, but it can be n-dimensional Euclidean space or more abstract spaces such as Banach spaces, where the concept of the difference between to points of the space can be defined [8].

9.5 Filtration

When the index set T has some total order relation, such in case of the index set being some subset of the real numbers, then it is possible to study the amount of information contained in a stochastic process X_t at $t \in T$, which can be interpreted as the moment or time t, by using the concept of a filtration [60, page 294 and 295][161, page 93]. A filtration, denoted here by $\{\mathcal{F}_t\}_{t \in T}$, on a probability space $(\Omega, \mathcal{F}, P)$ is a family of sigma-algebras such that $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for all $s \leq t$, where $t, s \in T$, so a filtration is an increasing sequence of sigma-algebras [60, page 294 and 295]. The intuition behind a filtration $\mathcal{F}_t$ is that as time t passes, more and more information on X_t is known or available, which is captured in $\mathcal{F}_t$, resulting in finer and finer partitions of Ω [95, page 23][117, page 37].

9.6 Modifications

A stochastic process X that has the same index set T, set space S, and probability space $(\Omega, \mathcal{F}, P)$ as another stochastic process Y is said to be a modification of Y if for all $t \in T$ the following

$$P(X_t = Y_t) = 1,$$

holds. Two stochastic processes that are modifications of each other are said to be stochastically equivalent or equivalent [31, page 530].

Instead of modification, the term version is also used [95, page 48] [119, page 14][60, page 472][4, page 14], however some authors use the term version when two stochastic processes have the same finite-dimensional distributions, but they may be defined on different probability spaces, so two stochastic processes that are versions of each other are also modifications of each other, in the latter sense, but not the converse [125, pages 18 and 19].

If a continuous-time real-valued stochastic process meets certain moment conditions on its increments, then the Kolomogorov continuity theorem says that there exists a modification of this process that has continuous sample paths with probability one, so the stochastic process has a continuous modification or version [119, page 14][60, page 472][7, page 20]. The theorem can also be derived for random fields so the index set is n-dimensional Euclidean space [101, page 31] as well as to stochastic processes with metric spaces as their state spaces [87, page 35].

9.7 Indistinguishable

Two stochastic processes X and Y defined on the same probability space $(\Omega, \mathcal{F}, P)$ with the same index set T and set space S are said be indistinguishable if the following

$$P(X_t = Y_t \text{ for all } t \in T) = 1,$$

holds [126, page 130][62, page 571]. If two X and Y are modifications of each other and are almost surely continuous, then X and Y are indistinguishable [85, page 11].

9.8 Separability

Separability is a property of a stochastic process based on its index set in relation to the probability measure. The property is assumed so that functionals of stochastic processes or random fields with uncountable index sets can form random variables. For a stochastic process to be separable, in addition to other conditions, its index set must be a separable space [8], which means that the index set has a dense countable subset [83, page 32][4, pages 14 and 15].

More precisely, a real-valued continuous-time stochastic process X with a probability space $(\Omega, \mathcal{F}, P)$ is separable if its index set T has a dense countable subset $U \subset T$ and there is a set $\Omega_0 \subset \Omega$ of probability zero, so $P(\Omega_0) = 0$, such that for every open set $G \subset T$ and every closed set $F \subset \mathbf{R} = (-\infty, \infty)$, the two events $\{X_t \in F \text{ for all } t \in G \cap T\}$ and $\{X_t \in F \text{ for all } t \in G\}$ differ at most from each other on a subset of Ω_0 [66, page 150][154, page 19][115, page 340].

The definition of separability [9] can also be stated for other index sets and state

[8] The term "separable" appears twice here with two different meanings, where the first meaning is from probability and the second from topology and analysis. For a stochastic process to be separable (in a probabilistic sense), its index set must be a separable space (in a topological or analytic sense), in addition to other conditions [143, page 94].

[9] The definition of separability for a continuous-time real-valued stochastic process can be stated in other ways [23, pages 526 and 527][31, page 535].

spaces [71, page 22], such as in the case of random fields, where the index set as well as the state space can be n-dimensional Euclidean space [4, pages 14 and 15][5, page 8].

The concept of separability of a stochastic process was introduced by Joseph Doob [83], where the underlying idea is to make a countable set of points of the index set determine the properties of the stochastic process [23, page 526]. Any stochastic process with a countable index set already meets the separability conditions, so discrete-time stochastic processes are separable [51, page 56].

A theorem by Doob, sometimes known as Doob's separability theorem, says that any real-valued continuous-time stochastic process has a separable modification [83, page 33][154, page 20][93]. Versions of this theorem also exist for more general stochastic processes with index sets and state spaces other than the real line [143, page 93].

9.9 Skorokhod space

A Skorokhod or Skorohod space is a mathematical space of all the functions that are right-continuous with left limits, defined on some interval of the real line such as $[0,1]]$ or $[0,\infty)$, and take values on the real line or on some metric space [159, pages 78 and 79] [71, page 24][30, page 53]. Such functions are known as cdlg or cadlag functions, based on the French acronym *continue droite, limite gauche*, due to the functions being right-continuous with left limits [159, pages 78 and 79][95, page 4], so they are also occasionally called corol functions (continuous on the right with left limits on the left) [13, page 29]. A Skorokhod function space is often denoted with the letter D [159, pages 78 and 79] [71, page 24][30, page 53][95, page 4], so the function space is also referred to as space D[9, page 420][22, page 121][159, 78]. The notation of the space can also include the interval on which all the cdlg functions are defined, so, for example, $D[0,1]$ denotes the space of cdlg functions defined on $[0,1]$ [22, page 121][95, page 4].

Skorokhod function spaces are frequently used in the theory of stochastic processes because it often assumed that the sample functions of continuous-time stochastic processes belong to a Skorokhod space [9, page 420][30, page 53]. Such spaces contain continuous functions, which correspond to sample functions of the Wiener process. But the space also has functions with discontinuities, which means that the sample functions of stochastic processes with jumps, such as the Poisson process (on the real line), are also members of this space [22, page 121][26, page 154].

10 Mathematical construction

In mathematics, constructions of mathematical objects are needed, which is also the case for stochastic processes, to prove that they exist mathematically [129, page 177]. There are two main approaches for constructing a stochastic process. One approach involves considering a measurable space of functions, defining a suitable measurable mapping from a probability space to this measurable space of functions, and then deriving the corresponding finite-dimensional distributions [4, page 13].

Another approach involves defining a collection of random variables to have specific finite-dimensional distributions, and then using Kolmogorov's existence theorem [10] to prove a corresponding stochastic process exists [4, page 13][129, page 177]. The theorem, which is an existence theorem for measures on infinite product spaces [55, page 410], says that if any finite-dimensional distributions satisfy two conditions, known as consistency conditions, then there exists a stochastic process with those finite-dimensional distributions [129, pages 177 and 178].

10.1 Construction difficulties

When constructing continuous-time stochastic processes certain mathenatical difficulties arise, due to the uncountable index sets, which do not occur with discrete-time processes [99, page 63][93, page 153]. One problem is that is it possible to have more than one stochastic process with the same finite-dimensional distributions. This means that the distribution of the stochastic process does not, necessarily, specify uniquely the properties of the sample functions of the stochastic process [4, page 14][31, pages 529 and 530].

Another problem is that functionals of continuous-time process that rely upon an uncountable number of points of the index set may not be measurable, so the probabilities of certain events may not be consistent [83, page 32]. For example, the supremum of a stochastic process or random field is not necessarily a well-defined random variable [5, page 8][93, page 154]. For a continuous-time stochastic process X, other characteristics that depend on an uncountable number of points of the index set T include [83, page 32]:
– a sample function of a stochastic process X is a continuous function of $t \in T$;
– a sample function of a stochastic process X is a bounded function of $t \in T$; and
– a sample function of a stochastic process $X$is an increasing function of $t \in T$. To overcome these two difficulties, various assumptions and approaches are possible [9, page 408].

10.2 Resolving construction difficulties

One approach, pioneered by Doob, for avoiding construction difficulties is to assume that the stochastic process is separable [10, page 211]. Separability ensures

[10]The theorem has other names including Kolmogorov's consistency theorem [10], Kolmogorov's extension theorem [119, page 11] or the Daniell-Kolmogorov theorem [161, page 124].

that infinite-dimensional distributions determine the properties of sample functions by requiring that sample functions are essentially determined by their values on a dense countable set of points in the index set [5, page 14]. Furthermore, if a stochastic process is separable, then functionals of an uncountable number of points of the index set are measurable and their probabilities can be studied [5, page 14][83, page 32].

Another approach is possible, originally developed by Skorokhod and Kolmogorov [10, page 211], for a continuous-time stochastic process with any metric space as its state space. For the construction of such a stochastic process, it is assumed that the sample functions of the stochastic process belong to some suitable function space, which is usually the Skorokhod space consisting of all right-continuous functions with left limits. This approach is now more used than the separability assumption [9, page 408][65], but such a stochastic process based on this approach will be automatically separable [31, page 536].

Although less used, the separability assumption is considered more general because every stochastic process has a separable version [65]. It is also used when it is not possible to construct a stochastic process in a Skorokhod space [31, page 535]. For example, separability is assumed when constructing and studying random fields, where the collection of random variables is now indexed by sets other than the real line such as n-dimensional Euclidean space [5, page 8][163, page 5].

The term regularity is used when discussing and assuming such conditions for a stochastic process to resolve issues [31, pages 532 to 537][93, pages 148 to 165]. In other words, to study stochastic processes with uncountable index sets, it is assumed that the stochastic process adheres to some type of regularity condition such as the sample functions being continuous [154, page 22][159, page 79].

11 Further reading

There are many, many books that cover the very broad topic of stochastic processes. The original book, now considered a classic, was by Doob[51]. There are others with similar names such those by Parzen[120], Rosenblatt[128] and Lamperti [103]. Books with more focus on applications include the two volumes by Karlin and Taylor [89, 90] and the less mathematical introduction by Ross [130]

On more specific topics, the two-volume set by Rogers and Williams[126, 127] covers much information on martingales and Markov processes. Mörters and Peres[117] treat the Brownian motion process, while this process with stochastic calculus are covered in many works including those by Klebaner [95], Steele [146], Øksendal [119], Shreve[142], and Karatzas and Shreve [88]. Applebaum [8] introduces Levy processes in an article for general mathematicians, but he also covers them in deep detail in his book [7]. Levy processes are also the topics of the books by Bertoin [21] and Sato [133].

Random fields (and their relationship to geometry) are covered by Adler [4] and the sequel by Adler and Taylor [5]. Point processes and their relations to stochastic processes are covered in the two-volume reference by Daley and Vere-Jones [47, 48].

The respective articles by Cramér [46] and Meyer [114] are good pieces for the history of probability theory and stochastic processes. The article by Jarrow and Protter [84] covers the history of stochastic differential equations. Bingham has penned a series of interesting papers [24, 25, 27] on the history of probability and stochastic processes.

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