Notes on the Poisson point process

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Abstract

The Poisson point process is a type of random object in mathematics known as a point process. For over a century this point process has been the focus of much study and application. This survey aims to give an accessible but detailed account of the Poisson point process by covering its history, mathematical definitions in a number of settings, and key properties as well detailing various terminology and applications of the point process, with recommendations for further reading.

1 Introduction

In probability, statistics and related fields, a Poisson point process or a Poisson process (also called a Poisson random measure or a Poisson point field) is a type of random object known as a point process that consists of randomly positioned points located on some underlying mathematical space [51, 54]. The Poisson point process has interesting mathematical properties [51][54, pages xv-xvi], which has led to it being frequently defined in Euclidean space and used as a mathematical model for seemingly random objects in numerous disciplines such as astronomy [4], biology [62], ecology [84], geology [17], physics [74], image processing [10], and telecommunications [33]. For example, defined on the real line, it can be interpreted as a stochastic process [1] called the Poisson process [83, pages 7–8][2][70, pages 94–95], forming an example of a Lévy process and a Markov process [73, page ix] [2]. In this setting, the Poisson process plays

[1]Point processes and stochastic processes are usually considered as different objects, with point processes being associated or related to certain stochastic processes [20, page 194][18, page 3][83, pages 7–8][25, page 580]. Point processes are now considered types of random measures [42, page 2][54, pages 127–128][**?**, page 106].

an important role in the field of queueing theory, where it is used to model certain random events happening in time such as the arrival of customers at a store or phone calls at an exchange [52, page 1][85, pages 1–32].

Defined on the Euclidean plane, the Poisson point process, also known as a spatial Poisson process [7], may represent scattered objects such as transmitters and receivers in a wireless network [5, 32, 12], particles colliding into a detector[83, pages 52–53], or trees in a forest [82, page 51]. In this setting, the process is often used as a mathematical model and serves as a cornerstone in the related fields of spatial point processes [21, pages 459–459][7], stochastic geometry [16], spatial statistics [61] and continuum percolation theory [58]. The Poisson point process can be defined on more general and abstract mathematical spaces. Beyond applications, it is a subject of mathematical study in its own right [51][54], with it and the Brownian motion process being generally considered the most important objects in the field of probability [41, page 176][65, page 8][79, page 32].

The Poisson point process has the property that each point is stochastically independent from all the other points in the point process, which is why it is sometimes called a purely or completely random process [20, page 26]. Despite its wide use as a model of random phenomena representable as points, the inherent nature of the process implies that it does not adequately describe phenomena in which there is sufficiently strong interaction between the points [16, pages 35–36]. This has led sometimes to the overuse of the point process in mathematical models [51, page 62], and has inspired other point processes, some of which are constructed via the Poisson point process, that seek to capture such interaction [16, pages 35–36].

The Poisson process is named after French mathematician Siméon Denis Poisson because if a collection of random points in some space form a Poisson process, then the number points in a region of finite size is directly related to the Poisson distribution, but Poisson, however, did not study the process, which independently arose in several different settings [81][31][54, pages 272–273]. Norbert Wiener gave the first mathematical construction of a Poisson point process that is considered mathematically rigorous, and referred to the point process as "discrete chaos" and "Poisson chaos"[54, pages 273 and 288][42, page 636] .

The Poisson point process is defined with a single object, which, depending on the context, may be a constant [2], an integrable function or, in more general settings, a type of measure [54, page 19][16, pages 41 and 51]. If this object is a constant, then the resulting process is called a homogeneous [54, pages 58–59][51, page 13] or stationary [16, page 42]) Poisson point process. Otherwise, the parameter depends on its location in the underlying space, which leads to the inhomogeneous or nonhomogeneous Poisson point process [20, page 22]. The word 'point' is often omitted, but there are other Poisson processes, which can,

[2] In the definition, strictly speaking, the constant is multiplied by a Lebeguese measure, and then the product can be treated as a single object [54, pages 58–59].

instead of points, consist of more complicated mathematical objects such as lines, giving a Poisson line process. But then such a Poisson process can be in turn reinterpreted as a Poisson point process on a mathematical space of lines [51, pages 73–78].

2 History

2.1 Poisson distribution

Despite its name, the Poisson point process was not discovered or studied by the French mathematician Siméon Denis Poisson, and the name is cited as an example of Stigler's law [81][31]. The name stems from its inherent relation to the Poisson distribution, derived by Poisson as a limiting case of the binomial distribution [54, page 272], which describes the probability of the sum of n Bernoulli variables with probability p, often likened to the number of heads (or tails) after n flips of a biased coin with the probability of a head (or tail) occurring being p [30, page 47]. For some positive constant $\Lambda > 0$, as n increases towards infinity and p decreases towards zero such that the product $np = \Lambda$ is fixed, the Poisson distribution more closely approximates that of the binomial [30, page 61]. Poisson derived the Poisson distribution, published in 1841, by examining the binomial distribution in the limits of p (to zero) and n (to infinity) [20, pages 8–9][54, pages 272–273].

The Poisson distribution only appears once in all of Poisson's work [80], and the result was not well known during his time, even though over the following years a number of people would use the distribution without citing Poisson including Philipp Ludwig von Seidel and Ernst Abbe [20, pages 8–9][81]. The distribution would be studied years after Poisson at the end of the 19th century in a different setting by Ladislaus Bortkiewicz who did cite Poisson and used the distribution with real data to study the number of deaths from horse kicks in the Prussian army [27][67][20, pages 8–9].

2.2 Discovery of the Poisson point process

The early history of the Poisson point process is complex due to its various applications in numerous fields, with the early work being published in different languages and in different settings, with no standard terminology and notation used [81]. Consquently, there are a number of claims for early uses and discoveries of the Poisson point process [31][54, pages 272–273].

It has been proposed that an early use of the Poisson point process was by John Michell in 1767, a decade before Poisson was born. Michell was interested in the probability of a star being within a certain region of another star under the assumption that the stars were 'scattered by mere chance', and studied an example consisting of the six brightest stars in the Pleiades, without deriving the Poisson distribution. This work would later inspire the astronomer and mathematician Simon Newcomb in 1860 to independently derive the Poisson distribution by taking limit of the bionomial distribution [31].

On the non-negative half-line, the Poisson point process has been discovered several times. In 1844 Robert Leslie Ellis published a paper where he introduced a renewal equation and derived the gamma distribution for the special case of exponential waiting times, which is considered an early discovery of the Poisson point process [54, pages 272–273][42, page 635]. Lundberg discovered the Poisson compound process, a generalization of the Poisson process, by deriving an differential equation now known as the Kolmogrov forward equation [54, page 273][31][42, pages 635–636].

Working also in the setting of time, Agner Krarup Erlang derived in 1909 the Poisson distribution when developing a mathematical model for the number of incoming phone calls in a finite time interval. Erlang, not at the time aware of Poisson's earlier work, assumed that the number phone calls arriving in each interval of time were independent of each other, and then found the limiting case, which is effectively recasting the Poisson distribution as a limit of the binomial distribution [81]. In 1910, after physicists Ernest Rutherford and Hans Geiger conducted an experiment on counting the number of alpha particles, English mathematician Harry Bateman derived the Poisson probabilities as a solution to a family of differential equations, although Bateman acknowledged that the solutions had been previously solved by others [20, page 9].

Inspired by the construction of a haemocytometer by Carl Zeiss, Ernst Abbe studied the problem of the number of blood cells located in a region or volume of space, deriving the Poisson distribution in a paper published in the 1879 [77, page 77][20, page 8]. Work published years later in a 1895 shows that Abbe anticipated the generality of the Poisson point process [54, pages 273 and 281].

The Poisson point process continued to be used in numerous fields by engineers and various scientists [81]. For example, the Poisson point process and other point processes were used in the field of teletraffic engineering [36]. In 1922 Swedish Chemist and Nobel Laureate Theodor Svedberg proposed a model in which a spatial Poisson point process is the underlying point process in order to study how plants are distributed [40, page 13].

2.3 Birth of modern probability theory

In 1933 Andrei Kolmogorov published his book on the foundations of probability theory [3], in which Kolmogorov used measure theory to develop an axiomatic framework for probability theory. The publication of this book is now

[3] The book was titled *Grundbegriffe der Wahrscheinlichkeitsrechnung* and then later translated into English and published in 1950 as *Foundations of the Theory of Probability* [11].

widely considered to be the birth of modern probability theory, when the theories of probability and stochastic processes became parts of mathematics [11, 19]. Starting in the 1930s, a number of mathematicians studied the Poisson process, and important contributions were made by Andrey Kolmogorov, William Feller, Aleksandr Khinchin [81], and Paul Lévy [page viii][42] among others [50].

2.4 Mathematical construction

Motivated by statistical mechanics, Norbert Wiener published a paper in 1938 [86], in which he gave the first derivation and definition of a spatial Poisson process that is considered mathematically rigorous [54, pages 273 and 288]. Wiener used the terms 'discrete chaos' and 'Poisson chaos', writing [86, page 927]:

> The discrete or Poisson chaos which we have thus defined is the chaos of an infinite random shot pattern, or the chaos of the gas molecules in a perfect gas in statistical equilibrium according to the old Maxwell statistical mechanics.

Wiener's PhD student Broackway McMillan also worked on the Poisson point process, completing a PhD on the topic [57]. Wiener and Aurel Wintner then published a paper in 1943 titled 'The Discrete Chaos" [87], in which they were the first to define the Poisson point process on a general mathematical space [54, pages 273 and 288].

2.5 History of terms

Conrad Palm in his 1943 dissertation studied the Poisson and other point processes in the one-dimensional setting by examining them in terms of the statistical or stochastic dependence between the points [20, page 14]. In his work exists the first known recorded use of the term "point processes" as "Punktprozesse" in German [20, page 14][31].

In 1928 Thorton C. Fry, in his book on engineering applications of probability, used the term "Poisson law", but it is believed [31] he used it in two senses, referring to the Poisson probability distribution and the Poisson process. In a 1937 paper Joseph Doob called it 'the time series with Poisson distribution' and then in 1938 Wiener called it 'disrete or Poisson chaos' [31].

It has been claimed [81] that Feller was the first in print to refer to it as the "Poisson process" in a 1940 paper[22], although Feller also used the expression Poisson ensemble in the second volume of his book published 1974 [23, pages 9–10]. It has been remarked [31] that in 1940 Ove Lundberg also used the term Poisson process in his PhD dissertation, in which Feller was acknowledged as an influence on Lundberg [29, page ix]. It has been remarked that both Feller and Lundberg used the term as though it were well-known, implying it was already in spoken use [31]. Feller worked from 1936 to 1939 alongside Harald Crameér at Stockholm University, where Lundberg was a PhD student under Cramér who did not use the term "Poisson process" in a book by him, finished in 1936, but did in subsequent editions. The term has not been found in work of the same period by the Soviets Kolmogorov and Khinchin or American mathematician Joseph Doob, suggesting that it was not standard in the Soviet Union or the USA. Similarly, the term does not appear in British literature before 1948. All this has led to the speculation that the term "Poisson process" was coined sometime between 1936 and 1939 at the Stockholm University [31].

3 Two defining properties

The Poisson point process is one of the most studied mathematical objects in both the field of point processes and in more applied disciplines concerning random phenomena due to its convenient properties as a mathematical model as well as being mathematically interesting [51, Preface][54, pages xv-xvi]. It may be defined, studied and used in one dimension (on the real line) where it can be interpreted as a counting process or part of a queueing model [85, pages 1 and 9]; in higher dimensions such as the plane where it plays a role in stochastic geometry and spatial statistics [16, Chapter 2]; or on more abstract mathematical spaces [54, page 12]. Consequently, the notation and terminology used to define and study the Poisson point process and points processes in general vary according to the context [16, pages 49, 108–110][20, page 19][31]. Despite all this, the Poisson point process has two defining properties [54, page 19][42, page 70][51, page 11].

3.1 Defining property: Poisson number of points

The Poisson point process is related to the Poisson distribution, which implies that the probability of a Poisson random variable N is equal to n is given by:

$$P\{N = n\} = \frac{\Lambda^n}{n!} e^{-\Lambda}$$

where $n!$ denotes n factorial and Λ is the single Poisson parameter that is used to define the Poisson distribution. If a Poisson point process is defined on some underlying mathematical space, called a "state space" [51, page 8][54, page 38] or "carrier space" [35, page 323][15], then the number of points in a bounded region of the space will be a Poisson random variable with some parameter, known as the intensity or mean measure of the Poisson point process (see Section11), and its form will depend on the setting [51, pages 11–14].

3.2 Defining property: Complete independence

The other defining property is that for a collection of disjoint (or non-overlapping) bounded subregions of the underlying space, the number of points of the Poisson point process in each bounded subregion will be completely independent from all the other points of the point process. This property is known under different names such as "complete randomness", "complete independence", [20, pages 26–27] or "independent scattering" [16, page 41][61, page 16][83, pages 33–34] and is common to all Poisson point processes (and Poisson processes in general). In other words, there is a lack of interaction between different regions and the points in general [61, page 17][16, page 44], which motivates the Poisson process being sometimes called a purely or completely random process [20, page 26].

3.3 Definition

The two properties [4] of the Poisson distribution and complete independence define the Poisson propertie [54, page 19][51, page 11][42, page 70]. More specifically, a point process N defined on some measurable state space S is a Poisson point process if it adheres these two properties, and the parameter Λ is a s-finite measure, meaning Λ can be expressed as a countable sum of finite measures [54, page 19].

4 Homogeneous Poisson point process

If a Poisson point process has a constant parameter, say, λ, then it is called a homogeneous or stationary Poisson point process . The parameter, called rate or intensity, is related to the expected (or average) number of Poisson points existing in some bounded region [51, page 13][61, page 14]. In fact, the parameter is λ can be interpreted as the average number of points per some unit of length, area or volume, depending on the underlying mathematical space, hence it is sometimes called the mean density [20, page 20]; see Section 11.

4.1 Defined on the real line

To define omogeneous Poisson point process, one can first consider two real numbers a and b, where $a \leq b$, and which may represent points in time. If the points form or belong to a homogeneous Poisson process with parameter $\lambda > 0$,

[4] These two properties are not logically independent because complete independence requires the Poisson distribution, but not necessarily the converse; Kingman [51, Section 1.3]. In fact, for certain Poisson point processes, only the complete independence property is needed, which then in turns gives the Poisson distribution [42, pages 2–3].

then the probability of n points existing in the above interval $(a, b]$ is given by:

$$P\{N(a,b] = n\} = \frac{[\lambda(b-a)]^n}{n!} e^{-\lambda(b-a)},$$

where $N(a, b]$ denotes the random number of points of a homogeneous Poisson point process existing with values greater than a but less than or equal to b. In other words, $N(a, b]$ is a Poisson random variable with mean $\lambda(b - a)$. Furthermore, the number of points in any two subjoint intervals, such as $(a_1, b_1]$ and $(a_2, b_2]$, are independent of each other, and this extends to any finite number of subjoint intervals [20, pages 19–20]. In the queueing theory context, one can consider a point existing (in an interval) as an event, but this is different to the word event in the probability theory sense [5]. In this context, λ is the expected number of arrivals that occur per unit of time [85, pages 9–10][72, page 70], and it is sometimes called the rate parameter [20, page 82].

For a more formal definition of a stochastic process, such as a point process, one can use the Kolmogorov theorem, which essentially says a stochastic process is characterized (or uniquely defined) by its finite-dimensional distribution, which in this context gives the joint probability of some number of points existing in each disjoint finite interval [20, pages 19–20, 381][21, pages 25–31]. More specifically, let $N(a_i, b]$ denote the number of points of (a point process) happening in the half-open interval $(a_i, b_i]$, where the real numbers $a_i < b_i \leq a_{i+1}$. Then for some positive integer k, the homogeneous Poisson point process on the real line with parameter $\lambda > 0$ is defined with the finite-dimensional distribution [20, pages 19–20]:

$$P\{N(a_i,b_i] = n_i, i = 1, \ldots, k\} = \prod_{i=1}^{k} \frac{[\lambda(b_i - a_i)]^{n_i}}{n_i!} e^{-\lambda(b_i - a_i)},$$

4.1.1 Key properties

The above definition has two important features pertaining to the Poisson point processes in general:

– the number of points in each finite interval has a Poisson distribution.

– the number of points in disjoint intervals are independent random variables.

Furthermore, it has a third feature related to just the homogeneous Poisson process:

–the distribution of each interval only depends on the length $b_i - a_i$, hence they are stationary (the process is sometimes called the stationary Poisson process). In other words, for any finite $t > 0$, the random variable $N(a+t, b+t]$ is independent of t [20, pages 19–20].

[5] It is possible for an event not happening in the queueing theory sense to be an event in the probability theory sense.

4.1.2 Law of large numbers

The quantity $\lambda(b_i - a_i)$ can be interpreted as the expected or average number of points occurring in the interval $(a_i, b_i]$, namely:

$$E\{N(a_i, b_i]\} = \lambda(b_i - a_i),$$

where E denotes the expectation operator. In other words, the parameter λ of the Poisson process coincides with the density of points. Furthermore, the homogeneous Poisson point process adheres to its own form of the (strong) law of large numbers [51, pages 41–44]. More specifically, with probability one:

$$\lim_{t\to\infty} \frac{E\{N(0,t]\}}{t} = \lambda,$$

where $\lim$ denotes the limit of a function.

4.1.3 Memoryless property

The distance between two consecutive points of a point process on the real line will be an exponential random variable with parameter λ (or equivalently, mean $1/\lambda$). This implies that the points have the memoryless property: the existence of one point existing in a finite interval does effect not the probability (distribution) of other points existing[85, pages 2–4].

4.1.4 Relationship to other processes

On the real line, the Poisson point process is a type of continuous-time Markov process known as a birth-death process (with just births and zero deaths) and is called a pure [72, page 235] or simple birth process [64, page 648]. More complicated processes with the Markov property, such as Markov arrival processes, have been defined where the Poisson process is a special case [68].

4.1.5 Counting process interpretation

The homogeneous Poisson point process, when considered on the positive half-line, is sometimes defined as a counting process , which can be written as $\{N(t), t \geq 0\}$ [72, pages 59–60][85, page 2]. A counting process represents the total number of occurrences or events that have happened up to and including time t. A counting process is the Poisson counting process with rate $\lambda > 0$ if it has three the properties:

– $N(0) = 0$;

– has independent increments; and

– the number of events (or points) in any interval of length t is a Poisson random variable with parameter (or mean) λt.

The last property implies

$$E[N(t)] = \lambda t.$$

The Poisson counting process can also be defined by stating that the time differences between events of the counting process are exponential random variables with mean $1/\lambda$ [85, page 2]. The time differences between the events or arrivals are known as interrarrival [72, page 64] or interoccurence times [85, page 2]. The times when these events occur or the locations of the points are called arrival times [7, page 3][33, page 22].

4.1.6 Martingale characterization

On the real line, the homogeneous Poisson point process has a connection to the theory of martingales via the following characterization: a point process is the homogeneous Poisson point process if and only if

$$N(-\infty, t] - t,$$

is a martingale [59].

4.1.7 Restricted to the half-line

If the homogeneous Poisson point process is considered on just on the half-line $[0, \infty)$, which is often the case when t represents time, as it does for the previous counting process [72, pages 59–60][85, page 2], then the resulting process is not truly invariant under translation [51, page 39]. In that case the process is no longer stationary, according to some definitions of stationarity [16, page 42][18, page 13].

4.1.8 Applications

There have been many applications of the homogeneous Poisson point process on the real line in an attempt to model seemingly random and independent events occurring. It has a fundamental role in queueing theory, which is the probability field of developing suitable stochastic models to represent the random arrival and departure of certain phenomena. For example, customers arriving and being served or phone calls arriving at a phone exchange can be both studied with techniques from queueing theory. [52][72, Chapter 2][85, Chapter 1].

4.1.9 Generalizations

The Poisson counting process or, more generally, the homogeneous Poisson point process on the real line is considered one of the simplest stochastic processes for counting random numbers of points [79, page 41][18, page 3]. The process can be generalized in a number of ways [20, page 111]. One possible generalization is to extend the distribution of interarrival times from the exponential distribution to other distributions, which introduces the stochastic process known as a renewal process[72, page 98]. Another generalization is to define it on higher dimensional spaces such as the plane [43, page 398].

4.2 Spatial Poisson point process

A spatial Poisson process is a Poisson point process defined on the plane $\mathbf{R}^2$ [59]. For its definition, consider a bounded, open or closed (or more precisely, Borel measurable) region B of the plane. Denote by $N(B)$ the (random) number of points of N existing in this region $B \subset \mathbf{R}^2$. If the points belong to a homogeneous Poisson process with parameter $\lambda > 0$, then the probability of n points existing in B is given by:

$$P\{N(B) = n\} = \frac{(\lambda|B|)^n}{n!} e^{-\lambda|B|}$$

where $|B|$ denotes the area of B.

More formally, for some some finite integer $k \geq 1$, consider a collection of disjoint, bounded Borel (measurable) sets $B_1, \ldots, B_k$. Let $N(B_i)$ denote the number of points of existing in B_i. Then the homogeneous Poisson point process with parameter $\lambda > 0$ has the finite-dimensional distribution [20, page 31]

$$P\{N(B_i) = n_i, i = 1, \ldots, k\} = \prod_{i=1}^{k} \frac{(\lambda|B_i|)^{n_i}}{n_i!} e^{-\lambda|B_i|}.$$

4.2.1 Applications

The Poisson point process has also been frequently used to model seemingly disordered spatial configurations of certain wireless communication networks[5][32][12]. In the 1960s, Edgar Gilbert developed a mathematical model of a wireless networks based on a Poisson point process, which is now considered the birth of percolation theory [24, pages 2 and 15].

Models of cellular or mobile phone networks have been developed where it is assumed the phone network transmitters, known as base stations, are positioned according to a homogeneous Poisson point process. Often the aim behind these networks models is to derive mathematical expressions describing a quantity known as the signal-to-interference-plus-noise ratio [5][32][13][12]. Another application of these mathematical models is the analysis of data routing algorithms in various types of wireless networks [47][6], which has been extended to studying the flow of certain items, such as messages, in random networks [38].

4.3 Defined in higher dimensions

The previous homogeneous Poisson point process immediately extends to higher dimensions by replacing the notion of area with (high dimensional) volume. For some bounded region B of Euclidean space $\mathbf{R}^d$, if the points form a homogeneous Poisson process with parameter $\lambda > 0$, then the probability of n points existing in $B \subset \mathbf{R}^d$ is given by:

$$P\{N(B) = n\} = \frac{(\lambda |B|)^n}{n!} e^{-\lambda |B|}$$

where $|B|$ now denotes the n-dimensional volume of B. Furthermore, for a collection of disjoint, bounded Borel sets $B_1, \dots, B_k \subset \mathbf{R}^d$, where $N(B_i)$ denotes the number of points of N existing in B_i. Then the corresponding homogeneous Poisson point process with parameter $\lambda > 0$ has the finite-dimensional distribution [20, pages 19–20]:

$$P\{N(B_i) = n_i, i = 1, \dots, k\} = \prod_{i=1}^{k} \frac{(\lambda |B_i|)^{n_i}}{n_i!} e^{-\lambda |B_i|}.$$

Homogeneous Poisson point processes do not depend on the position of the underlying state space through its parameter λ, which implies it is both a stationary process (invariant to translation) and an isotropic (invariant to rotation) stochastic process [16, page 42]. Similarly to the one-dimensional case, the homogeneous point process is restricted to some bounded subset of $\mathbf{R}^d$, then depending on some definitions of stationarity, the process is no longer stationary [51, page 39][16, page 42].

4.4 Points are uniformly distributed

If the homogeneous point process is defined on the real line as a mathematical model for occurrences of some phenomenon, then it has the characteristic that the positions of these occurrences or events on the real line (often interpreted as time) will be uniformly distributed. More specifically, if an event occurs (according to this process) in an interval $(a - b]$ where $a \leq b$, then its location will be a uniform random variable defined on that interval [20, pages 21–22][pages 15–16]itetijms2003first. Furthermore, the homogeneous point process is sometimes called the uniform Poisson point process (see Section 11). This uniformity property extends to higher dimensions in the Cartesian coordinate systems, but it does not hold in other coordinate systems such as the polar coordinate system [20, page 25][16, pages 53–54].

5 Inhomogeneous Poisson point process

The inhomogeneous or nonhomogeneous Poisson point process (see Section 11 for terminology) is a Poisson process with Poisson paramter defined as some location-dependent function in the underlying space on which the Poisson process is defined. For Euclidean space R^d, this is achieved by introducing a locally integrable positive function $\lambda(x)$, where x is a n-dimensional located point in $\mathbf{R}^d$, such that for any bounded region B the (d-dimensional) volume integral of $\lambda(x)$ over region B is finite. In other words, if this integral, denoted by $\Lambda(B)$, is [61, page 13]:

$$\Lambda(B) = \int_B \lambda(x) dx < \infty,$$

where dx is a (d-dimensional) volume element [6], then for any collection of disjoint bounded Borel sets $B_1, \dots, B_k$, an inhomogeneous Poisson process with (intensity) function $\lambda(x)$ has the finite-dimensional distribution [20, page 31]:

$$P\{N(B_i) = n_i, i = 1, \dots, k\} = \prod_{i=1}^{k} \frac{(\Lambda(B_i))^{n_i}}{n_i!} e^{-\Lambda(B_i)}.$$

Furthermore, $\Lambda(B)$ has the interpretation of being the expected number of points of the Poisson process located in the bounded region B, namely

$$\Lambda(B) = E[N(B)].$$

5.1 Defined on the real line

On the real line, the inhomogeneous or non-homogeneous Poisson point process has mean mean measure given by a one-dimensional integral. For two real numbers a and b, where $a \leq b$, denote by $N(a, b]$ the number points of an inhomogeneous Poisson process with intensity function $\lambda(t)$ with values greater than a but less than or equal to b. The probability of n points existing in the above interval $(a, b]$ is given by[20, pages 34–35]

$$P\{N(a, b] = n\} = \frac{[\Lambda(a, b)]^n}{n!} e^{-\Lambda(a,b)}.$$

where the $\Lambda(a, b)$:

$$\Lambda(a, b) = \int_a^b \lambda(t) dt,$$

[6] Instead of $\lambda(x)$ and dx, one could write, for example, in (two-dimensional) polar coordinates $\lambda(r, \theta)$ and $r dr d\theta$, where r and θ denote the radial and angular coordinates respectively, and so dx would be an area element in this example.

which is called is the intensity or mean measure (see Section11). This implies that the random variable $N(a, b]$ is a Poisson random variable with mean $E\{N(a, b]\} = \Lambda(a, b)$.

A feature of the one-dimension setting considered useful is that an inhomogeneous Poisson point process can be made homogeneous by a monotone transformation, which is achieved with the inverse of Λ [51, page 21].

5.1.1 Counting process interpretation

The inhomogeneous Poisson point process, when considered on the positive half-line, is also sometimes defined as a counting process . With this interpretation, the process, which is sometimes written as $\{N(t), t \geq 0\}$, represents the total number of occurrences or events that have happened up to and including time t. A counting process is said to be an inhomogeneous Poisson counting process if it has the four properties [72, pages 59–60][85, page 2]:

– $N(0) = 0$;
– has independent increments;
– $P\{N(t+h) - N(t) = 1\} = \lambda(t)h + o(h)$; and
– $P\{N(t+h) - N(t) = 2\} = o(h)$,

where $o(h)$ is asymptotic notation for $o(h)/h \to 0$ as $h \to 0$.

The above properties imply that $N(t+h) - N(t)$ is a Poisson random variable with the parameter (or mean)

$$E[N(t+h) - N(t)] = \int_t^{t+h} \lambda(s)ds,$$

which implies

$$E[N(t)] = \int_0^t \lambda(s)ds.$$

5.2 Spatial Poisson point process

An inhomogeneous Poisson process, just like a homogeneous Poisson process, defined on the plane $\mathbf{R}^2$ is called a spatial Poisson point process [7]. Calculating its intensity measure requires performing an area integral of its intensity function over some region. For example, its intensity function (as a function of Cartesian coordinates x and y) may be

$$\lambda(x, y) = e^{-(x^2+y^2)},$$

hence it has an intensity measure given by the area integral

$$\Lambda(B) = \int_B e^{-(x^2+y^2)} dxdy,$$

where B is some bounded region in the plane R^2. The previous intensity function can be re-written, via a change of coordinates, in polar coordinates as

$$\lambda(r, \theta) = e^{-r^2},$$

which reveals that the intensity function in this example is independent of the angular coordinate θ, or, in other words, it is isotropic or rotationally invariant. The intensity measure of the Poisson point process is then given by the integral

$$\Lambda(B) = \int_B e^{-r^2} r dr dd\theta,$$

where B is some bounded region in the plane R^2 [54, pages 65–66].

5.3 In higher dimensions

In the plane, $\Lambda(B)$ corresponds to an area integral while in $\mathbf{R}^d$ the integral becomes a (d-dimensional) volume integral.

5.4 Applications

The inhomogeneous process is used in the fields of counting processes and in queueing theory because the real line is often interpreted as time [72, pages 78–28][85, pages 22–24]. Examples of phenomena that have been represented by or appear as an inhomogeneous Poisson point process include:

– Goals being scored in a soccer game [37].

– Defects in a circuit board [39]

– Power values of wireless signals when the signals experience sufficiently strong and random fading or shadowing [46, 48, 71].

In addition to direct applications, two different inhomogeneous Poisson point processes defined on the non-negative numbers are used to construct two types of the Poisson-Dirichlet process [34], which are used in various applications such Bayesian statistics, number theory, population genetics [66, page 856] or wireless networks [45].

On the plane, the Poisson point process is of fundamental importance in the related disciplines of stochastic geometry [16] and spatial statistics [7][61]. This point process is not stationary owing to the fact that its distribution is dependent on the location of underlying space or state space. Hence, it can be used to model phenomena with a density that varies over some region. In other words, the phenomena can be represented as points that have a location-dependent density. Uses for this point process as a mathematical model are diverse and have appeared across various disciplines including the study of salmon and sea lice in the oceans [53], forestry [84], and naval search problems [56].

5.5 Interpretation of the intensity function $\lambda(x)$

The Poisson intensity function $\lambda(x)$ has an interpretation with the volume element dx in the infinitesimal sense: $\lambda(x)dx$ is the infinitesimal probability of a point of a Poisson point process existing in a region of space with volume dx located at x [16, page 51].

For example, given a homogeneous Poisson point process on the real line, the probability of finding a single point of the process in a small interval of width δx is approximately $\lambda\delta x$. Such intuition is how the Poisson point process is sometimes introduced and its distribution derived[18, pages 3–4].

5.6 Simple point process

If a Poisson point process has an intensity measure that is a diffuse (or non-atomic) Radon measure, then it is a simple point process. For a simple point process, the probability of a point existing at a single point or location in the underlying (state) space is either zero or one. This implies that, with probability one, no two (or more) points of a Poisson point process coincide in location in the underlying space [16, page 37][32, page 25].

6 Statistical estimation

7 Simulation

Simulating a Poisson point process on a computer is done in a bounded region of space, known as a simulation window, and requires two steps: appropriately creating a random number of points and then suitably placing the points in a random manner. Both these two steps depend on the specific Poisson point process that is being simulated [83, pages 13–16][16, pages 53–55].

7.1 Step 1: Number of points

The number of points N in the simulation window, denoted here by W, needs to be simulated, which is done by using a (pseudo)-random number generating algorithm capable of simulating Poisson random variables.

7.1.1 Homogeneous case

For the homogeneous case with the constant λ, the mean of the Poisson random variable N is set to $\lambda|W|$ where $|W|$ is the length, area or (d-dimensional) volume of W.

7.1.2 Inhomogeneous case

For the inhomogeneous case, $\lambda|W|$ is replaced with the (d-dimensional) volume integral

$$\Lambda(W) = \int_W \lambda(x)dx$$

7.2 Step 2: Positioning of points

The second step requires randomly placing the N points in the window W.

7.2.1 Homogeneous case

For the homogeneous case, all points are uniformly and independently placed in the (interval) window W. For higher dimensions in a Cartesian coordinate system, each coordinate is uniformly and independently placed in the window W. If the window is not a subspace of Cartesian space (for example, inside a unit sphere or on the surface of a unit sphere), then the points will not be uniformly placed in W, and suitable change of coordinates (from Cartesian) are needed [16, page 53–55].

7.2.2 Inhomogeneous case

For the inhomogeneous, a couple of different methods can be used depending on the nature of the intensity function $\lambda(x)$ [83, pages 13–16][16, page 53–55]. If the intensity function is sufficiently simple, then independent and random non-uniform (Cartesian or other) coordinates of the points can be generated. For example, simulating a Poisson point process on a circular window can be done for an isotropic intensity function (in polar coordinates r and θ), implying it is rotationally variant or independent of θ but dependent on r, by a change of variable in r if the intensity function is sufficiently simple[16, page 53–55].

For more complex intensity functions, one can use an acceptance-rejection method, which consists of using (that is, accepting) only certain random points and not using (that is, rejecting) the other points, based on the ratio [83, pages 13–16]

$$\frac{\lambda(x_i)}{\Lambda(W)} = \frac{\lambda(x_i)}{\int_W \lambda(x)dx.}$$

where x_i is the point under consideration for acceptance or rejection.

8 Poisson point process on general spaces

Instead of the Euclidean space, the Poisson point process can be defined on an arbitrary measurable space. On this mathematical space, it is defined in relation to a s-finite measure Λ [54, page 19], which means the measure Λ is a countable sum of finite measures [54, page 244][42, page 21][26, page 83]. Then a point process N is a Poisson point process on the space S with intensity Λ if it has the two following properties [54, pages 16 and 19]:

– The number of points in a bounded Borel set $B \subset S$ is a Poisson random variable with mean $\Lambda(B)$. In other words, denote the total number of points located in B by $N(B)$, then the probability that the random variable $N(B)$ is equal to n is given by:

$$P\{N(B) = n\} = \frac{(\Lambda(B))^n}{n!} e^{-\Lambda(B)}$$

–the number of points in n disjoint Borel sets forms n independent random variables. The above Poisson process is sometimes referred to a general Poisson point process [16, Page 42][32, page 18][7]

The measure Λ keeps its interpretation of being the expected number of points of N located in the bounded region B[54, page 19], namely

$$\Lambda(B) = E[N(B)].$$

Furthermore, if the measure Λ is absolutely continuous such that it has a density (that is, a Radon–Nikodym density or derivative) with respect to the Lebesgue measure [54, page 244], then for all Borel sets B it can be written as [16, page 51]:

$$\Lambda(B) = \int_B \lambda(x) dx,$$

where the density $\lambda(x)$ is known, among other terms, as the intensity function; see Section 11 for terminology.

9 Poisson random measure

Applebaum page 90; see sato too

since random measures occur everywhere in our discipline and play a fundamental role in practically every area of stochastic processes.Kallenberg measures page vii

CITATION ; see Terminology

"fundamental role of Poisson processes for the theory of Lévy processes"[41, page 177]

[7]Some definitions use a Radon measure for the intensity of a general Poisson process[16, Page 42][32, page 18], but more recent probability books use a s-finite measure [54, page 244][42, page 21].

10 Relation to other random objects

Kallenberg measures : Apart from the importance of Poisson processes to model a variety of random phenomena, such processes also form the basic building blocks for construction of more general processes, similar to the role of Brownian motion in the theory of continuous processes...Cluster processes of various kinds, in their turn, play a fundamental role within the theory of branching processes.

Bernoulli process

Poisson-Dirichlet process

Gaussian fields; Alder

11 Terminology

In addition to the word point often being omitted, the terminology of the Poisson point process and point process theory varies, which has been criticized [31]. The homogeneous Poisson (point) process is also called a stationary Poisson (point) process [20, pages 19–20], sometimes the uniform Poisson (point) process [51, page 13], and in the past it was, by William Feller and others, referred to as a Poisson ensemble of points [23, pages 9–10][69]. The term "point process" has been criticized and some authors prefer the term "random point field" [16, page xviii], hence the terms "Poisson random point field" or "Poisson point field" are also used [60]. A point process is considered, and sometimes called, a random counting measure [**?**, page 106], hence the Poisson point process is also referred to as a "Poisson random measure"[73, page 144], a term used in the study of Lévy processes [73, **?**], but some choose to use the two terms for slightly different random objects [**?**].

The inhomogenous Poisson point process, as well as be being called "non-homogeneous" [20, page 22] or "non-homogeneous" [18, page 4], is sometimes referred to as the "non-stationary" [85, page 22], "heterogeneous" [55][9][78] or "spatially dependent" Poisson (point) process [53][63].

The measure Λ is sometimes called the "parameter measure" [20, page 34], "intensity measure", "first moment measure" [16, pages 45 and 51] or mean measure [51, page 12]. If Λ has a derivative or density, denoted by $\lambda(x)$, it may be called the intensity function of the general Poisson point process [16, page 51] or simply the "rate" or "intensity" [51, page 13]. For the homogeneous Poisson point process, the intensity is simply a constant $\lambda > 0$, which can be referred to as the "mean rate', "mean density" [20, page 20] or "density"[16, pages xxii and 51]. For $\lambda = 1$, the corresponding process is sometimes referred to as the "standard" Poisson (point) process [28][61, page 14].

The underlying mathematical and measurable space on which the point process, Poisson or other, is defined is known as a "state space"[8] [51, page 8][54,

[8]In the context of point processes, the term "state space" means the space on which the point

page 38] or "carrier space" [35, page 323][15].

12 Notation

The notation of the Poisson point process varies partly due to the theory of point processes, which has a couple of mathematical interpretations. For example, a simple Poisson point process may be considered as a random set, which suggests the notation $x \in N$, implying that x is a random point belonging to or being an element of the Poisson point process N. Another, more general, interpretation is to consider a Poisson or any other point process as a random counting measure, so one can write the number of points of a Poisson point process N being found or located in some (Borel measurable) region B as $N(B)$, which is a random variable. These different interpretations results in notation being used from mathematical fields such as measure theory and set theory [16, pages 110–111].

For general point processes, sometimes a subscript on the point symbol, for example x, is included so one writes (with set notation) $x_i \in N$ instead of $x \in N$, and x can be used in integral expressions such as Campbell's theorem, instead of denoting random points [5]. Furthermore, sometimes an uppercase letter denotes the point process, while a lowercase denotes a point from the process, so, for example, the point x (or x_i) belongs to or is a point of the point process X, or with set notation, $x \in X$ [61]. Furthermore, the integral (or measure theory) and set theory notation can be used interchangeably. For example, for a point process N defined on the Euclidean state space $\mathbf{R}^d$ and a (measurable) function f on $\mathbf{R}^d$, the expression

$$\int_{\mathbf{R}^d} f(x)N(dx) = \sum_{x_i \in N} f(x_i),$$

demonstrates two different ways to write a summation over a point process [16, pages 110–111].

Another reason for the various notation is the setting of the Poisson point process. For example, on the real line, the Poisson process can be interpreted as a counting process, and the notation $\{N(t), t \geq 0\}$ is used to to represent the Poisson process [85, pages 9–10][72, page 70].

13 Results

13.1 Restriction theorem

[51, page 17] [54, page 39]

process is defined such as the real line[51, page 8][61, Preface], which corresponds to the index set in stochastic process terminology.

13.2 Avoidance function

The avoidance function [20, page 135][51, page 134] or void probability [16, page 138] v of a point process N is defined in relation to some set B, which is a subset of the underlying space on which the point process is defined, as the probability of no points of N existing in the set B. More precisely, for a test set B, the avoidance function is given by [16, page 138]:

$$v(B) = P(N(B) = 0).$$

In the case of a Poisson point process N with intensity measure Λ, the avoidance function is [54, page 50][51, page 134]:

$$v(B) = e^{-\Lambda(B)}.$$

13.2.1 Rényi's theorem

Any simple point process, meaning the probability of two or more points of the point process coinciding in location on the underlying space is zero [16, page 37][32, page 25], is completely characterized by their void probabilities [54, page 50]. Rényi's theorem says that if a simple point process N with intensity measure Λ has the void probability :

$$P(N(B) = 0) = v(B) = e^{-\Lambda(B)}$$

where B is a Borel set of the underlying space on which the Poisson point process is defined. Then N is a Poisson point process with intensity measure Λ [54, page 50]. The result is named Alfréd Rényi who discovered it in the case of a homogeneous point process in one-dimension [51, page 34].

13.3 Contact distribution function

For a Poisson point process N on $\mathbf{R}^d$ with intensity measure Λ the contact distribution function is:

$$H(r) = 1 - e^{-\Lambda(b(o,r))},$$

where $|b(o, r)|$ denotes the volume (or more specifically, the Lebesgue measure) of the (hyper) ball of radius r. For the homogeneous, the contact distribution function becomes

$$H(r) = 1 - e^{-\lambda|b(o,r)|}.$$

In the plane $\mathbf{R}^2$, this expression becomes

$$H(r) = 1 - e^{-\lambda\pi r^2}.$$

13.4 Nearest neighbour function

For a Poisson point process N on $\mathbf{R}^d$ with intensity measure Λ the nearest neighbour function is:

$$D_x(r) = 1 - e^{-\Lambda(b(x,r))},$$

where $|b(x,r)|$ denotes the volume (or the Lebesgue measure) of the (hyper) ball of radius r. For the homogeneous case, the nearest neighbour function becomes

$$D_x(r) = 1 - e^{-\lambda|b(x,r)|}.$$

In the plane $\mathbf{R}^2$ with the reference point located at the origin denoted by o, this becomes

$$D_o(r) = 1 - e^{-\lambda\pi r^2}.$$

13.4.1 J-function

For the homogeneous Poisson point process, the spherical distribution function and nearest neighbour function are identical, which can be used to statistically test if a point process data appears to be that of a Poisson process. For example, in spatial statistics the J-function is defined for all $r \geq 0$ as [16]:

$$J(r) = \frac{1 - D_o(r)}{1 - H(r)}$$

For a homogeneous Poisson point process, the J function is simply $J(r) = 1$.

13.5 Laplace functionals

The Laplace functional [16, 20] of a point process N is defined as [5]:

$$L_N = E[e^{-\int_{\mathbf{R}^d} f(x)N(dx)}],$$

where f is any measurable non-negative function on $\mathbf{R}^d$ and

$$\int_{\mathbf{R}^d} f(x)N(dx) = \sum_{x_i \in N} f(x_i).$$

The Laplace functional can be used to prove various results about certain point processes [5, 20].

For a Poisson point process N with intensity measure Λ, the Laplace functional is a consequence of Campbells theorem [51] and is given by [5]:

$$L_N = e^{-\int_{\mathbf{R}^d}(1-e^{f(x)})\Lambda(dx)},$$

which for the homogeneous case is:

$$L_N = e^{-\lambda\int_{\mathbf{R}^d}(1-e^{f(x)})dx}.$$

13.6 Probability generating functionals

The probability generating function of non-negative integer-valued random variable leads to the probability generating functional being defined analogously. For a point process N defined on $\mathbf{R}^d$, the probability generating functional is defined as [16, page 125][54, page 25]

$$G(v) = E\left[\prod_{x\in N} v(x)\right]$$

where v is any non-negative bounded function v on $\mathbf{R}^d$ such that $0 \leq v(x) \leq 1$. The product in the above expression is performed for all the points in N. If the intensity measure Λ of N is locally finite, then the G is well-defined for any measurable function v on $\mathbf{R}^d$ [16, page 125].

For a Poisson point process N with intensity measure Λ the generating functional is given by [16, page 126]:

$$G(v) = e^{-\int_{\mathbf{R}^d}[1-v(x)]\Lambda(dx)},$$

which in the homogeneous case reduces to

$$G(v) = e^{-\lambda\int_{\mathbf{R}^d}[1-v(x)]dx}.$$

14 Point process operations

Mathematical operations can be performed on point processes. The operations can also be used to create new point processes, which are then also used as mathematical models for the random placement of certain objects [16][5].

One example of an operation is known as thinning, which entails deleting or removing the points of some point process according to a rule, hence creating a new process with the remaining points (the deleted points also form a point process). Another example is superimposing (or combining) point processes into one point process.

Under suitable conditions, when an operation is performed on a Poisson point process, the operation produces another (usually different) Poisson point process, demonstrating an aspect of mathematical closure [51].

CITATION

14.1 Thinning

For the Poisson process, the independent $p(x)$-thinning operations results in another Poisson point process. More specifically, a $p(x)$-thinning operation applied to a Poisson point process with intensity measure Λ gives a point process of removed points that is also Poisson point process N_p with intensity measure Λ_p, which for a bounded Borel set B is given by:

$$\Lambda_p(B) = \int_B p(x)\Lambda(dx)$$

Furthermore, after randomly thinning a Poisson point process, the kept or remaining points also form a Poisson point process, which has the intensity measure

$$\Lambda_p(B) = \int_B (1 - p(x))\Lambda(dx).$$

The two separate Poisson point processes formed respectively from the removed and kept points are stochastically independent of each other. In other words, if a region is known to contain n kept points (from the original Poisson point process), then this will have no influence on the random number of removed points in the same region. [51][16]

14.2 The Mecke equation

An important concept in the theory point processes is Palm conditioning, which forms the foundation of Palm calculus. Intuitively speaking, Palm conditioning involves considering a point process N conditioned on a point of the point process existing in a known location x of the underlying state space S. This point process N with a point x is called the Palm version of N, and it can be written as $N + \delta_x$, where δ_x is a Dirac measure, which represents a point existing at x.

CITATION

The Mecke equation or Mecke's formula provides a way to compute the expectation of integrals with respect to a Poisson point process, where the integrand can depend on both a single point process and all the points of the Poisson point process. In this sense, it an extension of Campbell's theorem for sums, where the integrand depends only on a single point process of the point process. But the Mecke equation also characterizes a Poisson point process, so it does not hold for other point processes.

More specifically, for a Poisson point process N defined on some measurable space S, such as Euclidean space $\mathbf{R}^d$, the Mecke equation is

$$\mathbf{E}[\int_S f(x, N)N(dx)] = \int_S \mathbf{E}[f(x, N + \delta_x)]\Lambda(dx),$$

where f is a non-negative measure function on $S\times$. A point process N is a Poisson point process with intensity measure Λ if and only if the Mecke equation holds for all functions f.

CITATION

14.3 Superposition

If there is a countable collection of point processes $N_1, N_2 \ldots$, then their superposition, or, in set theory language, their union

$$N = \bigcup_{i=1}^{\infty} N_i,$$

also forms a point process. In other words, any points located in any of the point processes $N_1, N_2 \ldots$ will also be located in the superposition of these point processes N.

14.4 Superposition theorem

The superposition theorem of the Poisson point process, which stems directly from the complete independence property, says [51] that the superposition of independent Poisson point processes $N_1, N_2 \ldots$ with mean measures $\Lambda_1, \Lambda_2, \ldots$ will also be a Poisson point process with mean measure

$$\Lambda = \sum_{i=1}^{\infty} \Lambda_i.$$

In other words, the union of two (or countably more) Poisson processes is another Poisson process. If a point x is sampled from a countable n union of Poisson processes, then the probability that the point x belongs to the jth Poisson process N_j is given by:

$$P(x \in N_j) = \frac{\Lambda_j}{\sum_{i=1}^{n} \Lambda_i}.$$

14.4.1 Homogeneous case

In the homogeneous case with constant $\lambda_1, \lambda_2 \ldots$, the two previous expressions reduce to

$$\lambda = \sum_{i=1}^{\infty} \lambda_i,$$

and

$$P(x \in N_j) = \frac{\lambda_j}{\sum_{i=1}^{n} \lambda_i}.$$

14.5 Clustering

The operation of clustering is performed when each point x of some point process N is replaced by another (possibly different) point process. If the original process N is a Poisson point process, then the resulting process N_c is called a Poisson cluster point process.

14.6 Random displacement

A mathematical model may require randomly moving points of a point process to other locations on the underlying mathematical space, which gives rise to a point process operation known as displacement [51] or translation [21]. The Poisson point process has been used to model, for example, the movement of plants between generations, owing to the displacement theorem, which says that the random independent displacement of points of a Poisson point process (on the same underlying space) forms another Poisson point process, provided some conditions on the random displacement [51].

14.6.1 Displacement theorem

One version of the displacement theorem [51] entails first considering a Poisson point process N on $\mathbf{R}^d$ with intensity function $\lambda(x)$. It is then assumed the points of N are randomly displaced somewhere else in $\mathbf{R}^d$ so that each points displacement is independent and that the displacement of a point formerly at x is a random vector with a probability density $\rho(x,\cdot)$[9]. Then the new point process N_D is also a Poisson point process with intensity function

$$\lambda_D(y) = \int_{\mathbf{R}^d} \lambda(x)\rho(x,y)dx,$$

which for the homogeneous case with a constant $\lambda > 0$ means

$$\lambda_D(y) = \lambda.$$

In other words, after each random and independent displacement of points, the original Poisson point process still exists.

The displacement theorem can be extended such that the Poisson points are randomly displaced from one Euclidean space $\mathbf{R}^d$ to another Euclidean space $\mathbf{R}^{d}$, where $d \geq 1$ is not necessarily equal to d [5].

14.7 Mapping

A property that is considered useful is the ability to map a Poisson point process from one underlying space to another space [51, pages 17–18]. If the mapping (or transformation) adheres to some conditions, then the resulting mapped (or transformed) collection of points also form a Poisson point process. This result is sometimes called the Mapping theorem [51][83][30].

[9] Kingman [51] calls this a probability density, but in other resources this is called a probability kernel [5], which is a general object used in other areas of probability such as Markov chains [41].

14.8 Mapping theorem

The Mapping theorem involves a Poisson point process with mean measure Λ on some underlying space. If the locations of the points are mapped (that is, the point process is transformed) according to some function to another underlying space, then the resulting point process is also a Poisson point process but with a different mean measure Λ.

More specifically, one can consider a (Borel measurable) function f that maps a point process N with intensity measure Λ from one space S, to another space T in such a manner so that the new point process N has the intensity measure:

$$\Lambda(B) = \Lambda(f^{-1}(B))$$

with no atoms, where B is a Borel set and f^{-1} denotes the inverse of the function f. If N is a Poisson point process, then the new process N is also a Poisson point process with the intensity measure Λ.

15 Approximation with a Poisson point process

The tractability of the Poisson process means that sometimes it is convenient to approximate a non-Poisson point process with a Poisson one. The overall aim is to approximate the both number of points of some point process and the location of each point by a Poisson point process [14]. There a number of methods that can be used to justify, informally or rigorously, the approximating of random events or phenomena with suitable Poisson processes. Some methods involve deriving upper bounds on the probability metrics between the Poisson and non-Poisson point processes, while other methods are justified with less formal heuristics [3].

15.1 Clumping heuristic

One such method for approximating random events or phenomena with Poisson processes is called the "clumping heuristic" [1]. The general heuristic or principle involves using the Poisson point process (or Poisson distribution) to approximate events, which are considered rare or unlikely, of some stochastic process. In some cases these rare events are close to independent, hence a Poisson point process can be used. When the events are not independent, but tend to occur in clusters or "clumps, then if these clumps are suitably defined such that they are approximately independent of each other, then the number of clumps occurring will be close to a Poisson random variable [3] and the locations of the clumps will be close to a Poisson process [1].

15.2 Stein's method

Stein's method or the Stein-Chen method, a rigorous mathematical technique originally developed for approximating random variables such as Gaussian and Poisson variables, has also been developed and applied to stochastic processes such as point processes. Stein's method can be used to derive upper bounds on probability metrics, which give ways to quantify how different two random mathematical objects vary stochastically, for the Poisson and non-Poisson point processes [8][14]. Upper bounds on probability metrics such as total variation and Wasserstein distance have been derived [14].

Researchers have applied Stein's method to Poisson point processes in a number of ways [14], including using Palm theory [15]. Techniques based on Steins method have been developed to factor into the upper bounds the effects of certain point process operations such as thinning and superposition [75][76]. Stein's method has also been used to derive upper bounds on metrics of non-Poisson and Poisson-based processes such as the Cox point process, which is a Poisson process with a random intensity measure [14].

16 Convergence to a Poisson point process

In general, when a operation is applied to a general point process, the resulting process is usually not a Poisson point process. For example, if a point process, other than a Poisson, has its points randomly and independently displaced, then the process would not necessarily be a Poisson point process. However, under certain mathematical conditions for both the original point process and the random displacement, it has been shown via limit theorems that if the points of a point process are repeatedly displaced in a random and independent manner, then the finite-distribution of the point process will converge (weakly) to that of a Poisson point process [21].

Similar convergence results have been developed for thinning and superposition operations [21] that show that such repeated operations on point processes can, under certain conditions, result in the process converging to a Poisson point processes, provided a suitable rescaling of the intensity measure (otherwise values of the intensity measure would approach zero or infinity) . Such convergence work is directly related to the results known as the Palm–Khinchin[10] equations, which has its origins in the work of Conny Palm and Aleksandr Khinchin [21], and help explains why the Poisson process can often be used as a mathematical model of various random phenomena.

[10] Also spelt Palm–Khintchine in, for example, Point Processes by Cox and Isham [18]

17 Generalizations of Poisson point processes

The Poisson point process can be generalized in different ways and used as a mathematical model or studied as a mathematical object [20, page 111][54, pages 127].

17.1 Mixed Poisson point processes

A mixed Poisson point process is formed by replacing the intensity λ of a homogeneous Poisson point process [11] with a non-negative random variable [61, page 57][16, pages 166–167][44, page 7]. Consequently, every single sample or realization of this point process looks a sample of some homogeneous Poisson point process [16, pages 166–167][20, page 200].

17.2 Cox processes

A Poisson point process can be generalized by letting its intensity measure Λ also be random and independent of the underlying Poisson point process, which forms a Cox point process or doubly stochastic Poisson point process, which was introduced by David Cox in 1955 under the latter name [51, page 65]. The Cox point process is considered a natural and important generalization of the Poisson point process [54, pages 127]. It is a generalization of a mixed Poisson point process [20, page 169][54, pages 134].

The intensity measure may be a realization of random variable or a random field. For example, if the logarithm of the intensity measure is a Gaussian field, then the resulting process is known as a log-Gaussian Cox process, which was introduced in the 1990 [16, page 167].

In general, the intensity measure of a Cox process is a realization of a non-negative locally finite random measure. Cox processes exhibit a clustering of points, which can be shown mathematically to be larger than those of Poisson point processes. The generality and tractability of Cox processes has resulted in them being used as models in fields such as spatial statistics and materials science [16, page 166–168].

17.3 Marked Poisson processes

For a given point process, each random point of a point process can have a random mathematical object, known as a "mark", assigned to it. These marks can be as diverse as integers, real numbers, lines, geometrical objects or other point processes [61, page 26][16, page 116]. The pair consisting of a point of the point

[11] A mixed Poisson point process has also been defined more generally by replacing the intensity measure Λ of a Poisson process with a product of a non-negative random variable and a s-finite [54, page 134] or a σ-finite intensity measure [42, page 243].

process and its corresponding mark is called a marked point, and all the marked points form a marked point process [32, pages 164–165]. It is often assumed that the random marks are all independent of each other and identically distributed, which makes the process easier to work with, yet the mark of a point can still depend on its corresponding point [51, page 55].

17.3.1 Marking theorem

If a general point process is defined on some mathematical space and the random marks are defined on another mathematical space, then the marked point process is defined on the Cartesian product of these two spaces. For a marked Poisson point process with independent and identically distributed marks, the Marking theorem states that this marked point process is also a (non-marked) Poisson point process defined on the aforementioned Cartesian product of the mathematical spaces, which is not true for general point processes [51, page 55][32, page 172].

17.4 Compound Poisson processes

The compound Poisson process is formed by first adding random values or weights to each point of a Poisson point process defined on some underlying state space, which means it is a formed from a marked Poisson point process, where the marks form a collection of independent and identically distributed non-negative random variables. In other words, for each point of the original Poisson process, there is an independent and identically distributed non-negative random variable. The compound Poisson process is then formed from the sum of all the random variables corresponding to points of the Poisson process located in a some region of the underlying mathematical space, where the number of random variables or points is a Poisson random variable[20, pages 198-199][54, page 153].

For example, one can consider a marked Poisson point processes formed from a Poisson point process N (defined on, for example, $\mathbf{R}^d$) and a collection of independent and identically distributed non-negative marks $\{M_i\}$ such that for each point x_i of the Poisson process N there is a non-negative random variable M_i. The resulting compound Poisson process is then

$$C(B) = \sum_{i=1}^{N(B)} M_i,$$

where $B \subset \mathbf{R}^d$ is a Borel measurable set [20, pages 198-199][54, page 153][44, page 22]. Compound Poisson processes defined on the non-negative numbers are perhaps the best known examples of this point process[44, page 22].

17.5 Infinitely divisible point processes

A point process or, more generally, a random measure is said to be infinitely divisible if, for every k, it can be represented as the superposition of k independent, identically distributed point processes or random measures [21, page 87–88][42, page 88]. The Poisson and the compound Poisson point processes are both examples of infinitely divisible point processes [21, page 87–88][54, pages 124–125 and 165]. The definition of infinitely divisible point processes is, by analogy, related to the one used to define random variables with infinitely divisible probability distributions [42, page 17].

18 Further reading

The definite source on the Poisson point process is the monograph by Kingman [51], where he defines and studies the Poisson point process in a general setting, and details connections to it and completely random measures and one type of Poisson-Dirichlet process. A modern sequel to this book is the newly published set of lectures notes by Last and Penrose, which connects the Poisson point process to many areas [54].

Points processes are covered in general in the two-volume reference text by Daley and Vere-Jones [20][21]. The very challenging but influential book by Kerstan, Matthes and Mecke laid down the theoretical foundation for infinitely divisible point processes [49].

Detailed historical notes on the Poisson and related point processes are found in appendices of the recent works by Last and Penrose [54] as well as Kallenberg [42]. A history of point processes in general is given in the first chapter of the first volume by Daley and Vere-Jones [20]. Historical notes on point processes are also found in the article by Guttorp and Thorarinsdottir [31].

The standard text on stochastic geometry is Chiu Mecke, Kendall and Stoyan f [16] or the previous edition by Mecke, Kendall, and Stoyan[82]. See the published lecture notes by Baddeley [7] or the books by Karr [44] or Møller and Waagepetersen [61] for introductions to spatial point processes in the context of spatial statistics. Random measures, including point processes, are the subject of the new book by Kallenberg [42].

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