When do wireless network signals appear Poisson?

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A user receives signals from many transmitters. The signals are distorted by physical fading effects which are often modelled as random.

Objective: Describe the distribution of the point process of signal strengths experienced by a typical user.

Implications for wireless network design eg the positioning of transmitters.

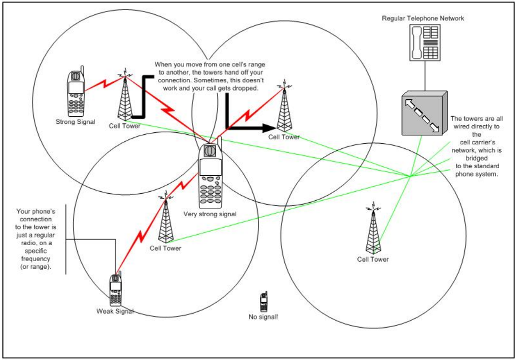


Figure : Lifted from `http://www.visiognomy.com/diagrams/archives/2005/02/16/cell-phone-towers/`

With a "typical user" located at the origin, the model has three components:

1. Transmitter positions : $\{x_i\}_{i\in\mathbb{N}} \subseteq \mathbb{R}^2/\{0\}$.
2. A path-loss or attenuation function: $\ell : \mathbb{R}^2/\{0\} \to (0,\infty)$.
3. Sequence of i.i.d. random variables representing fading effects (eg signals colliding with obstacles like buildings).

$$0 < S_1, S_2, \dots$$

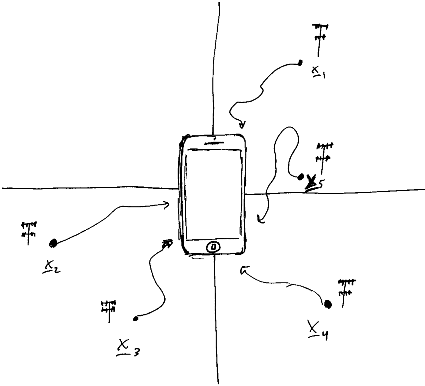


Figure : Sketch by N. Ross

Signal propagation model:

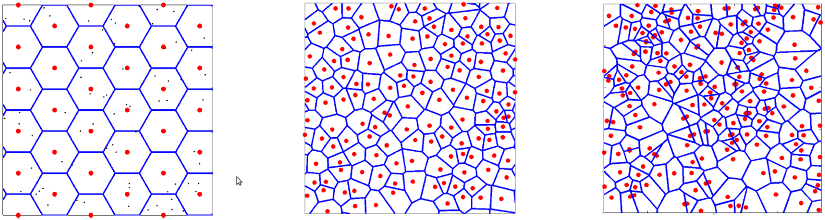
$$P_i = S_i \ell(x_i) = \frac{S_i}{g(x_i)}$$

where $g(x_i) := 1/\ell(x_i)$ is the path-gain function .

What is the random behaviour of power strengths $\{P_1, P_2, \dots\}$ or the propagation process $\{1/P_1, 1/P_2, \dots\}$?

Common assumptions

- Simple power-law: $g(x) = |x|^{\beta}$ for constant $\beta > 2$,
- In dense urban areas, S_i are often log-normally distributed, but can be exponentially or gamma distributed.
- Assume positions are random $\Phi = \{X_i\}$, usually a homogeneous Poisson process
- Often need Palm distribution and Laplace functional of the point process
- Recent work involves determinantal point processes to capture "repulsion" between transmitters



Grid Model — Actual 4G Deployment — Poisson Point Process (PPP)

Figure : Lifted from a talk by Harpreet S. Dhillon – for more pictures see 'Modeling and Analysis of K-Tier Downlink Heterogeneous Cellular Networks' by Dhillon et al., 2012.

Poisson transmitters implies Poisson signals

- Transmitters form a Poisson process $\Phi = \{X_i\}$ on $\mathbb{R}^2$ with density λ
- Define propagation process (inverse of power values P_i):

$$Z := \{Y_i\} \equiv \left\{ \frac{g(X_i)}{S_i} : X_i \in \Phi \right\}. \qquad (1)$$

- Definition based on convention ie the strongest signals are near zero
- Captures how the network "appear" to a user or observer.

Lemma (Just the mapping theorem)

Under the Poisson model with function $g(x) = |x|^\beta$ and random S such that $\mathbb{E}[S^{\frac{2}{\beta}}] < \infty$. Then the propagation process $Z = \{Y_i\}$ is an inhomogeneous Poisson point process on $\mathbb{R}_+$ with intensity measure

$$\Lambda_Z([0,t)) = at^{\frac{2}{\beta}}$$

where $a := \lambda\pi\mathbb{E}(S^{\frac{2}{\beta}})$.

Deterministic positioning of transmitters

- For $0 < \lambda < \infty$, assume a deterministic point pattern $\phi = \{x_i\}_i \subseteq \mathbb{R}^2/\{0\}$ of transmitters such that

$$\frac{\phi(r)}{\pi r^2} \to \lambda, \quad \text{as} \quad r \to \infty.$$

  where $\phi(r)$ denotes the number of points of ϕ within distance r of the origin ie number of points of ϕ in $B_0(r)$.
- Assume (rescaled) log-normal fading variables:

$$S_i^{(\sigma)} = e^{\sigma N_i - \sigma^2/\beta},$$

  where N_i are i.i.d. standard normal variables.
- Assume $g(x) = |x|^\beta$.
- Propagation process:

$$W^{(\sigma)} := \left\{ \frac{g(x_i)}{S_i^{(\sigma)}} : x_i \in \phi \right\} = \left\{ \frac{|x_i|^{\beta_i}}{S_i^{(\sigma)}} : x_i \in \phi \right\}.$$

Signals can “appear” Poisson under strong fading

Theorem (Blaszczyszyn, Karray, Keeler 2013, 2014)

Provided $g(x) = |x|^{\beta}$ and log-normal $S_i^{(\sigma)}$, then as $\sigma \to \infty$ (implying $S_i^{(\sigma)} \to 0$ in distribution), the point process $W^{(\sigma)} = \{Y_i^{(\sigma)}\}$ converges weakly to an inhomogeneous Poisson point process on $\mathbb{R}_+$ with intensity measure

$$\Lambda_W([0,t)) = a t^{\frac{2}{\beta}}$$

where $a := \lambda \pi \mathbb{E}([S_i^{(\sigma)}]^{\frac{2}{\beta}})$.

- Observed a couple of times via simulation in engineering literature.
- Proof uses classic translation convergence results eg Chapter 11 in Daley and Vere-Jones (2008).
- Relies heavily upon properties of $g(x) = |x|^{\beta}$ and log-normal S_i eg normal distribution is divisible and symmetric, $g^{-1}(S_i)$ is also log-normal.
- Can this convergence result be extended to more general $g(x)$ and S_i?
- Can bound be derived between $W^{(\sigma)}$ and a Poisson process with the same intensity measure?

Assumptions and notation for generalization

Transmitter positioning:

Let $\phi = \{x_i\}_i \subseteq \mathbb{R}^d/\{0\}$ be a locally finite collection of points in $\mathbb{R}^d$ such that

$$\frac{\phi(r)}{\pi r^2} \to \lambda, \quad \text{as} \quad r \to \infty. \tag{2}$$

Define $\mathcal{I}$ as a finite or countable index set such that $\phi = \{x_i : \; i \in \mathcal{I}\}$.

Path-gain function:

Let $g : \mathbb{R}^d \to \mathbb{R}_+$ be a positive Borel measurable mapping.

Fading variables:

Let $\{S_i : \; i \in \mathcal{I}\}$ be a sequence of i.d.d. positive random variables.

Propagation process:

Let $Y_i = g(x_i)/S_i$ and define the corresponding propagation point process

$$W := \{Y_i\}_{i \in \mathcal{I}}$$

Let $p_i(t) := \mathbb{P}(0 < Y_i \leq t)$ and $M(t) := \sum_{i \in \mathcal{I}} p_i(t)$ be the mean measure of W
Let Z be a Poisson process on $\mathbb{R}_+$ having a mean measure $M(t)$.

Approximation theorem: Bounds on total variation

- Recall the total variation distance between two probability measures ν_1, ν_2 on the same measurable space $(\mathcal{D}, \mathcal{F}(\mathcal{D}))$ is defined as

$$d_{\mathrm{TV}}(\nu_1, \nu_2) = \sup_{A \in \mathcal{B}(\mathcal{D})} |\nu_1(A) - \nu_2(A)|.$$

- Consider propagation point process W and Z restricted to compact domain $[0, t]$, denoted by $W|_t$ and $Z|_t$
- Denote the laws of $W|_t$ and $Z|_t$ by $\mathcal{L}(W|_t)$ and $\mathcal{L}(W|_t)$.

Theorem (Keeler, Ross, and Xia 2014)

Provided the previous conditions, then the following bounds hold

$$\frac{1 \wedge M(t)^{-1}}{32} \sum_{i \in \mathcal{I}} p_i(t)^2 \leq d_{TV}(\mathcal{L}(Z|_t), \mathcal{L}(W|_t)) \leq \sum_{i \in \mathcal{I}} p_i(t)^2 \leq M(t) \sup_{i \in \mathcal{I}} p_i(t).$$

- Proof of the term $\sum_i p_i(t)^2$ is due to a coupling argument (cf Le Cam's theorem).
- Far right-hand side stems from the definition of the mean measure $M(t) = \sum_i p_i(t)$; far left-hand side is due to Barbour and Hall (1984).

Convergence theorem

Theorem (Keeler, Ross, and Xia 2014)

$g : \mathbb{R}^d \to \mathbb{R}_+$ such that $g(x) = h(|x|)$ for a continuous and nondecreasing h .
$(S(\sigma))_{\sigma \geq 0}$ is a family of positive random variables indexed by some non-negative parameter σ.
$W^{(\sigma)}$ is the process generated by $S(\sigma)$, g and ϕ .
If

$$(i)\ S(\sigma) \xrightarrow{\mathbb{P}} 0 \quad \text{and} \quad (ii)\ \mathbb{E}[W^{(\sigma)}(t)] \to L(t),$$

as $\sigma \to \infty$, then $W^{(\sigma)}$ converges weakly to a Poisson process on $\mathbb{R}_+$ with mean measure L.

- Intuitively, most points of ϕ are being sent out to infinity in $W^{(\sigma)}$ because $S(\sigma)$ tends to zero, .
- Poisson limit is due to the thinning of the points in ϕ, but the retained points are redistributed
- Thinning scheme is different from the classical thinning schemes in the literature eg Kallenberg (1975), Brown (1979), Schuhmacher (2005, 2009).

A simple example with Bernoulli fading variables (by N. Ross)

Consider a point pattern ϕ such that the mapped points $\{g(x_i)\}_i$ are the positive integers $\{1, 2, ...\}$.
Divide each point i by a random variable $S_i(\sigma)$, hence the point process

$$1/S_1(\sigma), 2/S_2(\sigma), ...,$$

where the $S_i(\sigma)$ are i.i.d. taking only two possible values:

$$P(S(\sigma) = 1/\sigma) = 1 - P(S(\sigma) = \sigma) = 1 - 1/\sigma.$$

$S(\sigma)$ tends to zero in probability as σ goes to infinity and by computing directly $P(i/S_i(\sigma) \leq t)$, we can see that the number of points in the interval $(0, t]$ converges to a Poisson variable, since

$$\mathbb{E}[\# \text{ of points} \leq t] = \sum_i P(i/S_i(\sigma) \leq t) \to t \quad \text{as} \quad n \to \infty,$$

and

$$\sum_i P(i/S_i(n) \leq t)^2 \to 0 \quad \text{as} \quad n \to \infty.$$

The previous theorem implies that the point process $\{1/S_1(\sigma), 2/S_2(\sigma), ...\}$ converges to a (homogeneous) Poisson process as σ goes to infinity.

- Replace ϕ with a locally finite point process Φ independent of $\{S_i\}_{i\in\mathbb{N}}$.
- Define

$$M^{\Phi}(t) = \int_{\mathbb{R}^d} p_{(x)}(t)\Phi(dx),$$

where $p_{(x)}(t) = \mathbb{P}(0 < g(x)/S \leq t)$.
- Conditional on Φ, let Z be the Cox process directed by the measure M^{Φ}.

Theorem (Keeler, Ross, and Xia 2014)

For Φ, the following bounds hold

$$d_{TV}(\mathcal{L}(Z|_t), \mathcal{L}(W|_t)) \leq \mathbb{E}\int_{\mathbb{R}^d} p_{(x)}(t)^2\Phi(dx)$$

- For random Φ, an analogue of the previous convergence result is possible.
- Process may converge to a Cox process if the limit of its mean measure is random.
- When will it converge to a Poisson or Cox process?

Theorem (Keeler, Ross, and Xia 2014)

Assume that Φ *is a process on* $\mathbb{R}^d$ *with a locally finite mean measure* $\Lambda(r) := \mathbb{E}[\Phi(r)]$ *such that* $\lim_{r\downarrow 0}\Lambda(r) = 0$ *and as* $r \to \infty$*,*

$$\Lambda(r) \to \infty, \qquad \frac{\mathrm{Var}(\Phi(r))}{\Lambda(r)^2} \to 0. \tag{3}$$

Assume $g(x) = h(|x|)$*, where* h *is continuous, nondecreasing and positive on* $\mathbb{R}_+$*. If*

$$(i) S(\sigma) \xrightarrow{\mathbb{P}} 0 \quad \text{and} \quad (ii) \int_{\mathbb{R}^d} \mathbb{P}\left(0 < \frac{g(x)}{S(\sigma)} \leq t\right)\Lambda(dx) \to L(t), \tag{4}$$

as $\sigma \to \infty$ *, then* $W^{(\sigma)}$ *converges weakly to a Poisson process* Z^L *with mean measure* L*.*

Interesting questions

- For a given transmitter configuration, do some fading models induce a propagation point process significantly closer to Poisson than others?
- How do the results translate to functions of the point process?
- What statistical parameter estimation methods can be developed?
- Can the results be extended to models with short range (spatial) dependence between the fading variables?
- How can the results be generalized?

Thank you.

References:

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