Signal-to-interference ratio in wireless communication networks

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Historical overview: stochastic models of wireless networks

- Communication networks have long inspired stochastic models
- 1909: Erlang discovered the Poisson process in a teletraffic context
- 1961: Gilbert proposed a stochastic and purely geometric model for wireless networks, considered birth of continuum percolation
- 1980s: Engineers developed mostly geometric models based on Poisson process for certain wireless networks.
- 1990s: Zozi and Pupolin did some pioneering (but often forgotten or overlooked) work on a signal-to-interference ratio (SIR) model.
- 2000s: Engineers returned to developing geometric models for ad hoc networks such as sensor networks and mobile ad hoc networks
- 2001: Baccelli and Błaszczyszyn, motivated by information theory, introduced a SIR coverage model based on stochastic geometry
- 2005/6: Dousse and friends introduced and studied SIR percolation, extending the original model of Gilbert.
- 2010: Andrews Baccelli, and Ganti adapted the SIR model for mobile or cellular phone networks.
- Research field explodes: more and more engineering papers using stochastic geometry, motivated by denser phone networks and new technologies being deployed to handle YouTube etc traffic

Coverage models

- Classic models of wireless communication networks involve transmitters (and receivers) scattered over some region.
- Transmitter locations often form a homogeneous Poisson process $\Phi = \{X_i\}$ with density λ on the plane $\mathbb{R}^2$
- Boolean model: each transmitter $X_i \in \Phi$ has a circular transmission range (ie forms a disc)
- Each transmission radius can be a fixed constant or a radium variable
- Very geometrical or intuitive model

Classic Boolean model

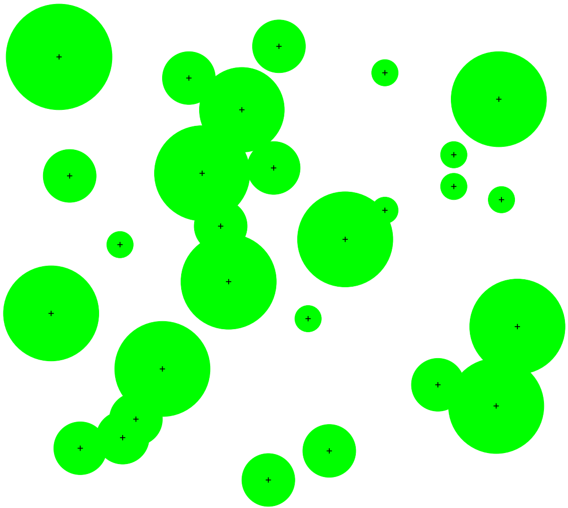
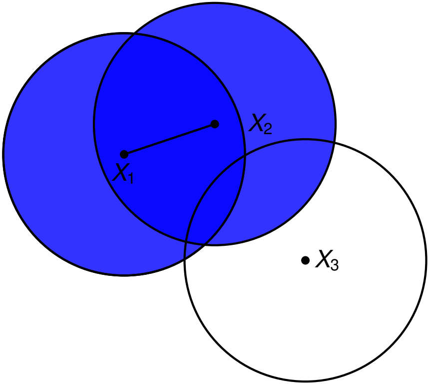


Figure : Green discs representing transmission ranges of transmitters in a wireless network. Picture by B. Błaszczyszyn

Birth of continuum percolation: Gilbert model

- 1961: Birth of continuum percolation with Gilbert's paper, which featured a model with fixed transmission discs.
- Create a undirected graph: two points X_1 and X_2 are connected if their discs overlap the centres of each other
- Gilbert showed a critical threshold (for density or disc radius) existed for the infinite or "giant" component



Another contender: Voronoi tesselation

$$\mathcal{C}_i = \{y \in \mathbb{R}^2 : |y - x| \leq |y - X_i| \ : \forall X_i \in \Phi\}$$

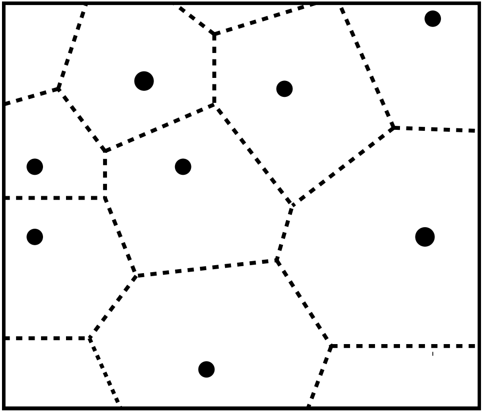


Figure : Picture by B. Błaszczyszyn

A new germ-grain model: the SIR coverage model

- Voronoi tessellation and Boolean Model are special cases of the general germ-grain coverage model.
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- General germ-grain coverage model $\{(X_i, \mathcal{C}_i)\}$
- $\{X_i\}$ are germs forming a point process Φ on $\mathbb{R}^2$
- $\mathcal{C}_i = \mathcal{C}_i(X_i, \Phi)$ are grains consisting of (possibly dependent) random closed subsets of $\mathbb{R}^2$.
- Often Poisson point process forms the "germs"
- Discs or Voronoi cells form the "grains" for the Boolean model or Voronoi tessellation respectively
- Baccelli and Błaszczyszyn (2001) introduced a new coverage model with dependent grains.
- SIR coverage model $\{(X_i, \mathcal{C}_i)\}$, where each $\mathcal{C}_i$ is a SIR cell formed by the SIR to be greater than some threshold τ
- SIR coverage model is also called shot-noise coverage model in Chiu, Stoyan, Kendall, Mecke (2013).

Signal-to-interference ratio (SIR)

- Wireless network of transmitters $\{X_i\}$ in $\mathbb{R}^2$ and a single (or "typical") user located at the origin o.
- R_i is the power received o of a signal originating from a transmitter at $X_i \in \mathbb{R}^2$.
- Performance metric is signal-to-interference-ratio (SIR) in relation to a base station at X_i

$$\text{SIR}(X_i, o) = \frac{R_i}{\sum_{j \neq i} R_j}, \qquad \text{SINR}(X_i, o) = \frac{R_i}{\sum_{j \neq i} R_j + N}$$

(Constant $N \geq 0$ is negligible in high density networks, so consider SIR).

- One important quantity: coverage probability of a single is defined as

$$\mathcal{P}_c(\tau) := \mathbb{P}(\max_i \text{SIR}(X_i, o) > \tau)$$

where τ is the (technology-dependent) SIR threshold

Standard models

- Standard network model: base stations $\{X_i\}$ positioned as a homogeneous Poisson process Φ with density λ on $\mathbb{R}^2$
- Standard propagation model:

$$R_i = F_i \ell(|X_i|) = \frac{F_i}{|X_i|^\beta},$$

- $\ell(x) = (|x|)^{-\beta}$ is the path-loss/attenuation function for constant $\beta > 2$
- Independent and identically-distributed (iid) random variables F_i represent propagation effects such as multi-path fading ie signals colliding with obstacles.
- F_i is often assumed to be exponentially or log-normally distributed
- Under this model, define the SINR cell as

$$\mathcal{C}_i = \left\{ y \in \mathbb{R}^2 : \frac{F_i \ell(|y - X_i|)}{\gamma \sum_{j \neq i} F_j \ell(|y - X_j|) + N} \geq \tau \right\}$$

where $0 \leq \gamma \leq 1$ is an "interference cancellation factor" – depends on the technology.

SIR cells for SIR threshold $\tau = 0.5$

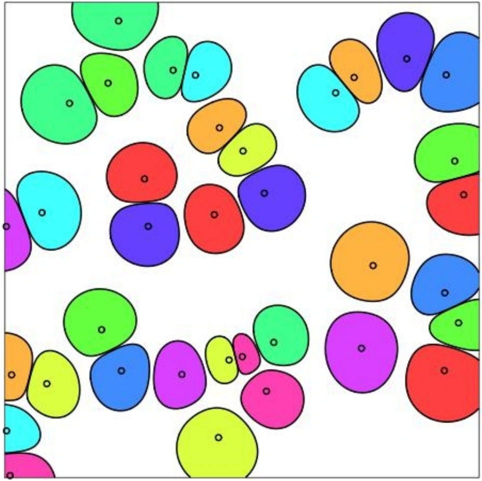

Figure : $\lambda = 10$, $\ell(r) = (Kr)^{-\beta}$, $K = 8000$, $\beta = 3$. Picture by B. Błaszczyszyn

SIR cells for SIR threshold $\tau = 0.4$

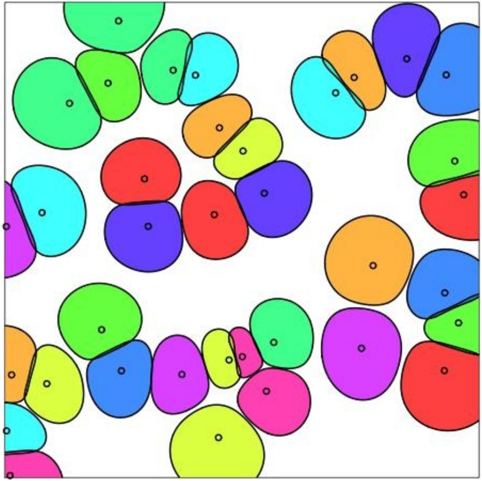

Figure : $\lambda = 10$, $\ell(r) = (Kr)^{-\beta}$, $K = 8000$, $\beta = 3$. Picture by B. Błaszczyszyn

SIR cells for SIR threshold $\tau = 0.3$

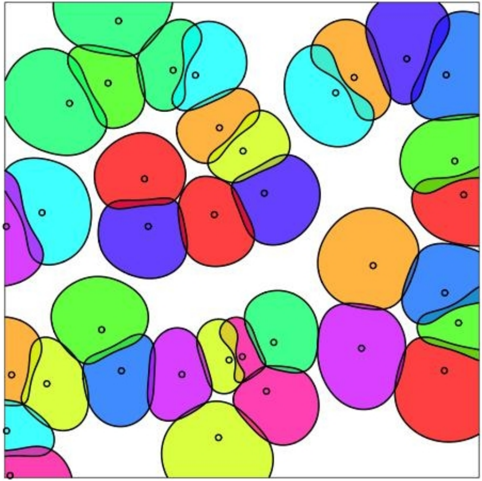

Figure : $\lambda = 10$, $\ell(r) = (Kr)^{-\beta}$, $K = 8000$, $\beta = 3$. Picture by B. Błaszczyszyn

SIR cells for SIR threshold $\tau = 0.2$

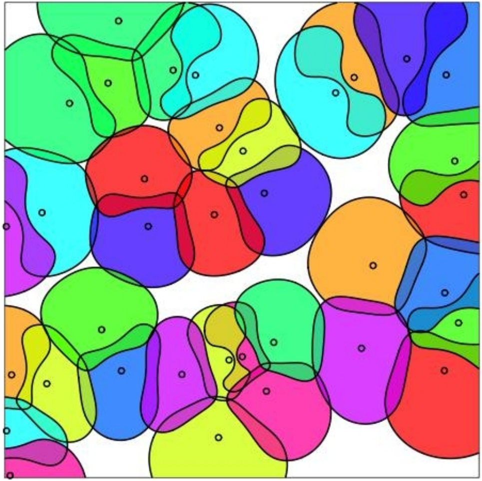

Figure : $\lambda = 10$, $\ell(r) = (Kr)^{-\beta}$, $K = 8000$, $\beta = 3$. Picture by B. Błaszczyszyn

SIR cells for SIR threshold $\tau = 0.1$

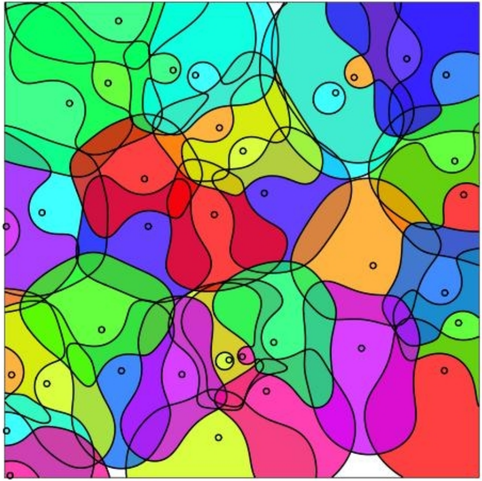

Figure : $\lambda = 10$, $\ell(r) = (Kr)^{-\beta}$, $K = 8000$, $\beta = 3$. Picture by B. Błaszczyszyn

SINR coverage model links some classic stochastic geometry models

Lemma

A single user can be covered by (at maximum) k transmitters in the entire network if $1/(k+1) \leq \tau < 1/k$

- SIR cell:

$$\mathcal{C}_i = \left\{ y \in \mathbb{R}^2 : \frac{F_i \ell(|y - X_i|)}{\gamma \sum_{j \neq i} F_j \ell(|y - X_j|) + N} \geq \tau \right\}$$

- When $\gamma = 0$ (no interference), the SINR cells are independent $\Rightarrow$ Boolean model approximations. Constant F_i gives Gilbert's disc model.
- When $N = 0$ (no noise) and $\beta \to \infty$, then SINR cells converge to Voronoi cells.
- Playing with $N \to 0$ and $\beta \to \infty$, the SINR model becomes the Johnson-Mehl model (for a simple Poisson birth process on $\mathbb{R}^2 \times [0, t]$).

Percolation results for the SIR coverage model

- Create a undirected graph: two points X_1 and X_2 are connected if at both points their respective SIRs exceed some threshold τ
- Dousse, Baccelli, and Thiran (2005) and Dousse, Franceschetti, Macris, Meester, and Thiran (2006). Also see monograph on SINR percolation by Franceschetti and Meester (2006)

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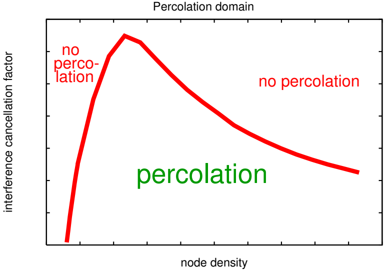

Figure : Path-loss model: $\ell(r) = \min(1, r^{-3})$. Plot by B. Błaszczyszyn

Increasing density λ may destroy the giant infinite component(s).

Tractable results for a single user: fading invariance

- Under standard Poisson model with singular path-loss function $\ell(r) = r^{-\beta}$, Poisson mapping theorem says that the signal power values

$$\Theta := \{R_i\} = \left\{ \frac{F_i}{|X_i|^\beta} : X_i \in \Phi \right\}$$

form an inhomogeneous Poisson point process on the positive real line with intensity measure

$$Q([t, \infty)) := \lambda \pi \mathbb{E}(F^{2/\beta}) t^{-2/\beta}$$

- Use exponential F and Laplace transforms, remove assumption
- To a single user, a deterministic (or even random non-Poisson) transmitter configuration $\phi = \{x_i\} \subset \mathbb{R}^2$ can still appear Poisson
- Keeler, Ross, and Xia (2014) showed that as iid $F_i \to 0$ (in distribution), the point process of power values converges to an inhomogeneous Poisson point process
- For a single user, define the *SIR point process* on the positive half-line $\mathbb{R}^+$ as

$$\{Z_i\} := \left\{ \frac{R_i}{I - R_i} : R_i \in \Theta \right\}, \tag{1}$$

where $I = \sum R_i^{-1}$ is total interference in the network.

Tractable results for a single user: k-coverage probability

Theorem (Błaszczyszyn, Karray, Keeler 2015)

Under a homogeneous Poisson model with density λ and singular path-loss function $\ell(r) = r^{-\beta}$, the k-coverage probability of a single user is

$$\mathcal{P}_c^{(k)}(\tau) = \sum_{n=k}^{\lceil 1/\tau \rceil} (-1)^{n-k} \binom{n-1}{k-1} \frac{(2/\beta)^{n-1}}{\tau_n^{(2n)/\beta} [C(\beta)]^n} \mathcal{J}_{n,\beta}(\tau_n) ,$$

where $\tau_n = \frac{\tau}{1-(n-1)\tau}$, $C(\beta) = \frac{2\pi}{\beta \sin(2\pi/\beta)}$ *, and for* $x \geq 0$ *,*

$$\mathcal{J}_{n,\beta}(x) = \int_{[0,1]^{n-1}} \frac{\prod_{i=1}^{n-1} v_i^{i(2/\beta+1)-1} (1-v_i)^{2/\beta}}{\prod_{i=1}^{n-1} [x + \eta_i(\{v_i\})]} dv_1 \ldots dv_{n-1} \qquad (2)$$

where $\eta_i(\{v_i\}) := (1 - v_i) \prod_{k=i+1}^{n-1} v_k$ *.*

$\mathcal{J}_{1,\beta}(x) = 1$ *so for* $\tau > 1$ *gives* $\mathcal{P}_c^{(1)}(\tau) = 1/[\tau^{2/\beta} C(\beta)]$
$\mathcal{J}_{2,\beta}(x)$ *is a sum of two hypergeometric functions* ${}_2F_1$

- Define the signal-to-total-interference ratio or *STIR process* on $(0, 1]$ as

$$\{Z_i'\} := \left\{ \frac{Y_i^{-1}}{I} : Y_i \in \Theta \right\}, \qquad Z_i = \frac{Z_i'}{1 - Z_i'}, \quad Z_i' = \frac{Z_i}{1 + Z_i} \tag{3}$$

- For parameters $0 \leq \alpha < 1$ and $\theta > -\alpha$, introduce a sequence of random variables $\tilde{V}_1 = U_1, \quad \tilde{V}_i = (1 - U_1) \dots (1 - U_{i-1}) U_i, \quad i \geq 2$, where U_1, U_2 are independent beta variables such that each U_i has $B(1 - \alpha, \theta + i\alpha)$ distribution. $\sum_{i=1}^{\infty} \tilde{V}_i = 1$ with probability one.
- Denote the decreasing order statistics of $\{\tilde{V}_i\}$ by $\{V_i\}$ such that $V_1 \geq V_2 \geq \dots$.
- Define the two-parameter Poisson-Dirichlet distribution with parameters α and θ, abbreviated as $\mathrm{PD}(\alpha, \theta)$, to be the distribution of $\{V_i\}$.
- See Pitman and Yor (1997) for interesting and useful results

Proposition (Błaszczyszyn and Keeler (2014))

The sequence $\{Z_i'\}$ is equal in distribution to $\{V_i\}$ for $\alpha = 2/\beta$ and $\theta = 0$. In other words, the STIR process $\{Z_i'\}$ is a $PD(2/\beta, 0)$ point process.

Summary and possible research directions

Summary:

- For information theoretic reasons, Baccelli and Błaszczyszyn (2001) introduced the SIR coverage model that bridges some well-known models
- Results exist on SIR (continuum) percolation by Dousse and friends.
- To a single user under *strong* and *independent* fading, networks appear Poisson or can be approximated with Poisson networks Keeler , Ross and Xia (2014)
- For single user and simple path-loss function, interesting SIR results exist, many observed independently in physics (eg Ruelle's cascade model) and mathematics (eg work of Pitman and Yor).

Research directions:

- Conditions for SINR model or purely geometric models.
- Study multiple users/receivers and multi-hop scenarios
- Dynamic situation with movement of transmitters and users
- Introduce dependence into fading variables eg Gaussian fields
- Use techniques from large-deviation theory to tackle the problem in the high density setting

Thank you.

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